Chapter 10. Infinite Sequences and Series

10.9 Convergence of Taylor Series

**Note.** We still have unanswered questions relevant to the generation of Taylor series from infinitely differentiable functions:

1. When does a Taylor series converge to its generating function?

2. How accurately do a function’s Taylor polynomials approximate the function on a given interval?

**Theorem 23. Taylor’s Theorem**

If \( f \) and its first \( n \) derivatives \( f', f'', \ldots, f^n \) are continuous on the closed interval between \( a \) and \( b \), and \( f^{(n)} \) is differentiable on the open interval between \( a \) and \( b \), then there exists a number \( c \) between \( a \) and \( b \) such that

\[
    f(b) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} + \frac{f^{(n+1)}(c)}{(n + 1)!}(x-a)^{n+1}.
\]
**Theorem. Taylor’s Formula**

If \( f \) is differentiable through order \( n + 1 \) in an open interval \( I \) containing \( a \), then for each \( x \in I \),

\[
f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} + R_n(x)
\]

where

\[
R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!} (x - a)^{n+1} \text{ for some } c \text{ between } a \text{ and } x.
\]

**Note.** If we can be insured that the remainder term \( R_n \) goes to 0 as \( n \to \infty \), then the Taylor series will converge to the generating function. This is summarized in the following theorem.

**Theorem 24. The Remainder Estimation Theorem.**

If there are positive constants \( M \) and \( r \) such that \( |f^{(n+1)}(t)| \leq Mr^{n+1} \) for all \( t \) between \( a \) and \( x \), inclusive, then the remainder term \( R_n(x) \) in Taylor’s Theorem satisfies the inequality

\[
|R_n(x)| \leq M \frac{r^{n+1}|x - a|^{n+1}}{(n + 1)!}.
\]

If these conditions hold for every \( n \) and all the other conditions of Taylor’s Theorem are satisfied by \( f \), then the series converges to \( f(x) \).

**Example.** Page 613 numbers 2, 16, and 46.