Chapter 8. Techniques of Integration

8.1 Integration by Parts

**Theorem. (Integration by Parts)** If \( u = u(x) \) and \( v = v(x) \) are differentiable functions of \( x \), then we have

\[
\int u \, dv = uv - \int v \, du.
\]

**Proof.** By the Product Rule we have

\[
\frac{d}{dx}[uv] = \left[ \frac{du}{dx} \right] v + u \left[ \frac{dv}{dx} \right].
\]

Integrating both sides with respect to \( x \) and rearranging leads to the integral equation:

\[
\int \left( \frac{du}{dx} \right) dx = \int \left( \frac{d}{dx}[uv] \right) dx - \int \left( \frac{dv}{dx} \right) dx
\]

\[
= uv - \int \left( \frac{dv}{dx} \right) dx.
\]

Q.E.D.
**Note.** Applying Integration by Parts to a definite integral, we have:

\[
\int_{v_1}^{v_2} u \, dv = (u_2 v_2 - u_1 v_1) - \int_{u_1}^{u_2} v \, du.
\]

In terms of areas, this gives the following figure.

![Figure 7.1 from Edition 10](image)

**Example.** Page 455 Example 2. Evaluate \( \int \ln x \, dx \).

**Example.** Page 456 Example 4. Evaluate

\[
\int e^x \cos x \, dx.
\]

(This is sort of weird!)
**Example.** Page 457 Example 5. Express $\int \cos^n x \, dx$ in terms of an integral of a lower power of $\cos x$. This is called a “reduction formula.”

**Examples.** Page 459 number 4, page 460 numbers 34 and 42, page 461 number 68.