Chapter 13. Vector-Valued Functions and Motion in Space

13.2. Integrals of Vector Functions; Projectile Motion

Definition. A differentiable vector function \( \mathbf{R}(t) \) is an antiderivative of a vector function \( \mathbf{r}(t) \) on in interval \( I \) if \( d\mathbf{R}/dt = \mathbf{r} \) at each point of \( I \). The indefinite integral of \( \mathbf{r} \) with respect to \( t \) is the set of all antiderivatives of \( \mathbf{r} \), denoted by \( \int \mathbf{r}(t) \, dt \). If \( \mathbf{R} \) is any antiderivative of \( \mathbf{r} \), then

\[
\int \mathbf{r}(t) \, dt = \{ \mathbf{R} \mid \mathbf{R}'(t) = \mathbf{r}(t) \} = \mathbf{R}(t) + C.
\]

Note. Whereas antiderivatives are functions, indefinite integrals are sets—indefinite integrals are sets of antiderivatives. We will use set notation sometimes, but often will abbreviate the set notation with the “\(+C\)” which is similar to how indefinite integrals were dealt with in Calculus 1. Also similar to Calculus 1, we see in the following definition that definite integrals are numbers.
Definition. If the components of \( \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \) are integrable over \([a, b]\), then so is \( \mathbf{r} \), and the definite integral of \( \mathbf{r} \) from \( a \) to \( b \) is

\[
\int_a^b \mathbf{r}(t) \, dt = \left( \int_a^b f(t) \, dt \right) \mathbf{i} + \left( \int_a^b g(t) \, dt \right) \mathbf{j} + \left( \int_a^b h(t) \, dt \right) \mathbf{k}.
\]

Examples. Page 738, number 4; and page 739, number 15.

Note. Suppose an object (a “projectile”) is given an initial velocity \( \mathbf{v}_0 \) and is then only acted on by the force of gravity (so we ignore frictional drag, for example). We assume that \( \mathbf{v}_0 \) makes an angle \( \alpha \) with the horizontal.

![Diagram](image)

Figure 13.10, page 735

Then

\[
\mathbf{v}_0 = (|\mathbf{v}_0| \cos \alpha)\mathbf{i} + (|\mathbf{v}_0| \sin \alpha)\mathbf{j} = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j}.
\]
Suppose the initial position is \( \mathbf{r}_0 = \mathbf{0} \). Newton’s Second Law of Motion says that the force acting on the projectile is equal to the projectile’s mass \( m \) times its acceleration (“\( F = ma \)”), or \( m(\frac{d^2\mathbf{r}}{dt^2}) \) where \( \mathbf{r} \) is the projectile’s position vector and \( t \) is time. With this gravitational force as the only force, \( -mg \mathbf{j} \), then

\[
m\frac{d^2\mathbf{r}}{dt^2} = -mg \mathbf{j} \quad \text{and} \quad \frac{d^2\mathbf{r}}{dt^2} = -g \mathbf{j}
\]

where \( g \) is the acceleration due to gravity. We find \( \mathbf{r} \) as a function of \( t \) by solving the initial value problem:

**Differential Equation:** \( \frac{d^2\mathbf{r}}{dt^2} = -g \mathbf{j} \)

**Initial Conditions:** \( \mathbf{r} = \mathbf{r}_0 \) and \( \frac{d\mathbf{r}}{dt} = \mathbf{v}_0 \) when \( t = 0 \).

We get by integration and use of initial conditions first that \( \frac{d\mathbf{r}}{dt} = -(gt)\mathbf{j} + \mathbf{v}_0 \) and then that \( \mathbf{r} = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t + \mathbf{r}_0 \). Expanding \( \mathbf{v}_0 \) and \( \mathbf{r}_0 \) gives

\[
\mathbf{r} = -\frac{1}{2}gt^2\mathbf{j} + (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{t}\mathbf{j} + \mathbf{0}
\]

or

\[
\mathbf{r} = (v_0 \cos \alpha)\mathbf{t}\mathbf{i} + ((v_0 \sin \alpha)t - \frac{1}{2}gt^2)\mathbf{j}.
\]

The angle \( \alpha \) is the projectile’s *launch angle* and \( v_0 \) is the projectile’s initial speed. As parametric equations, we have

\[
x = (v_0 \cos \alpha)t \quad \text{and} \quad y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2.
\]

**Example.** Page 739, number 22.
**Note.** We can easily find the the maximum height, range, and flight time of a projectile. We get:

- **Maximum Height:** $y_{\text{max}} = \frac{(v_0 \sin \alpha)^2}{2g}$
- **Flight Time:** $t = \frac{2v_0 \sin \alpha}{g}$
- **Range:** $R = \frac{v_0^2}{g} \sin 2\alpha$.

**Example.** Page 740, number 32.