Chapter II. Metric Spaces and the Topology of $\mathbb{C}$

II.1. Definitions and Examples of Metric Spaces—Proofs of Theorems
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Theorem II.1.13(c1). Let $X$ be a metric space and $A \subset X$. Then $\text{int}(A) = X \setminus (X \setminus A)^-$.

Proof. Let $x \in X \setminus (X \setminus A)^-$. Well, $(X \setminus A)^-$ is closed (it is the intersection of a collection of closed sets; use Theorem 1.11(c)). So $X \setminus (X \setminus A)^-$ is open and so (by the definition of open) there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subset X \setminus (X \setminus A)^-$. Hence $B(x; \varepsilon) \cap (X \setminus A)^- = \emptyset$, and so $B(x; \varepsilon) \subset A$. Since $B(x; \varepsilon)$ is open and a subset of set $A$, then $x \in \text{int}(A)$. So $X \setminus (X \setminus A)^- \subset \text{int}(A)$.

Let $x \in \text{int}(A)$. Then since $\text{int}(A)$ is open (it is the union of a collection of open sets; use Theorem 1.9(c)). So (by definition) there is $\varepsilon > 0$ such that $B(x; \varepsilon) \subset \text{int}(A)$. Now $X \setminus B(x; \varepsilon)$ is closed and $X \setminus A \subset X \setminus B(x; \varepsilon)$, so $(X \setminus A)^- \subset X \setminus B(x; \varepsilon)$ (the set on the right hand side is closed by definition) and $x \not\in (X \setminus A)^-$. Therefore $x \in X \setminus (X \setminus A)^-$ and so $\text{int}(A) \subset X \setminus (X \setminus A)^-$. Hence, $\text{int}(A) = X \setminus (X \setminus A)^-$. 
Theorem II.1.13(c1). Let $X$ be a metric space and $A \subset X$. Then
\[ \text{int}(A) = X \setminus (X \setminus A)^- . \]

Proof. Let $x \in X \setminus (X \setminus A)^-$. Well, $(X \setminus A)^-$ is closed (it is the intersection of a collection of closed sets; use Theorem 1.11(c)). So $X \setminus (X \setminus A)^-$ is open and so (by the definition of open) there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subset X \setminus (X \setminus A)^-$. 
Theorem II.1.13(c1). Let $X$ be a metric space and $A \subset X$. Then $\text{int}(A) = X \setminus (X \setminus A)^{-}$.

**Proof.** Let $x \in X \setminus (X \setminus A)^{-}$. Well, $(X \setminus A)^{-}$ is closed (it is the intersection of a collection of closed sets; use Theorem 1.11(c)). So $X \setminus (X \setminus A)^{-}$ is open and so (by the definition of open) there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subset X \setminus (X \setminus A)^{-}$. Hence $B(x; \varepsilon) \cap (X \setminus A)^{-} = \emptyset$, and so $B(x; \varepsilon) \cap (X \setminus A) = \emptyset$. Therefore $B(x; \varepsilon) \subset A$. 


Theorem II.1.13(c1). Let $X$ be a metric space and $A \subset X$. Then
\[ \text{int}(A) = X \setminus (X \setminus A)^-. \]

**Proof.** Let $x \in X \setminus (X \setminus A)^-$. Well, $(X \setminus A)^-$ is closed (it is the intersection of a collection of closed sets; use Theorem 1.11(c)). So $X \setminus (X \setminus A)^-$ is open and so (by the definition of open) there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subset X \setminus (X \setminus A)^-$. Hence $B(x; \varepsilon) \cap (X \setminus A)^- = \emptyset$, and so $B(x; \varepsilon) \cap (X \setminus A) = \emptyset$. Therefore $B(x; \varepsilon) \subset A$. Since $B(x; \varepsilon)$ is open and a subset of set $A$, then $x \in \text{int}(A)$. So $X \setminus (X \setminus A)^- \subset \text{int}(A)$. 
Theorem II.1.13(c1). Let $X$ be a metric space and $A \subset X$. Then \( \text{int}(A) = X \setminus (X \setminus A)^{-} \).

**Proof.** Let \( x \in X \setminus (X \setminus A)^{-} \). Well, \( (X \setminus A)^{-} \) is closed (it is the intersection of a collection of closed sets; use Theorem 1.11(c)). So \( X \setminus (X \setminus A)^{-} \) is open and so (by the definition of open) there exists \( \varepsilon > 0 \) such that \( B(x; \varepsilon) \subset X \setminus (X \setminus A)^{-} \). Hence \( B(x; \varepsilon) \cap (X \setminus A)^{-} = \emptyset \), and so \( B(x; \varepsilon) \cap (X \setminus A) = \emptyset \). Therefore \( B(x; \varepsilon) \subset A \). Since \( B(x; \varepsilon) \) is open and a subset of set \( A \), then \( x \in \text{int}(A) \). So \( X \setminus (X \setminus A)^{-} \subset \text{int}(A) \). Let \( x \in \text{int}(A) \).
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Let $x \in \text{int}(A)$. Then since $\text{int}(A)$ is open (it is the union of a collection of open sets; use Theorem 1.9(c)). So (by definition) there is $\varepsilon > 0$ such that $B(x; \varepsilon) \subset \text{int}(A)$. 


Theorem II.1.13(c1). Let $X$ be a metric space and $A \subseteq X$. Then $\text{int}(A) = X \setminus (X \setminus A)^{-}$.

**Proof.** Let $x \in X \setminus (X \setminus A)^{-}$. Well, $(X \setminus A)^{-}$ is closed (it is the intersection of a collection of closed sets; use Theorem 1.11(c)). So $X \setminus (X \setminus A)^{-}$ is open and so (by the definition of open) there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subset X \setminus (X \setminus A)^{-}$. Hence $B(x; \varepsilon) \cap (X \setminus A)^{-} = \emptyset$, and so $B(x; \varepsilon) \cap (X \setminus A) = \emptyset$. Therefore $B(x; \varepsilon) \subset A$. Since $B(x; \varepsilon)$ is open and a subset of set $A$, then $x \in \text{int}(A)$. So $X \setminus (X \setminus A)^{-} \subset \text{int}(A)$.

Let $x \in \text{int}(A)$. Then since $\text{int}(A)$ is open (it is the union of a collection of open sets; use Theorem 1.9(c)). So (by definition) there is $\varepsilon > 0$ such that $B(x; \varepsilon) \subset \text{int}(A)$. Now $X \setminus B(x; \varepsilon)$ is closed and $X \setminus A \subset X \setminus B(x; \varepsilon)$, so $(X \setminus A)^{-} \subset X \setminus B(x; \varepsilon)$ (the set on the right hand side is closed by definition) and $x \notin (X \setminus A)^{-}$. 
**Theorem II.1.13(c1).** Let $X$ be a metric space and $A \subset X$. Then $\text{int}(A) = X \setminus (X \setminus A)^-$. 

**Proof.** Let $x \in X \setminus (X \setminus A)^-$. Well, $(X \setminus A)^-$ is closed (it is the intersection of a collection of closed sets; use Theorem 1.11(c)). So $X \setminus (X \setminus A)^-$ is open and so (by the definition of open) there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subset X \setminus (X \setminus A)^-$. Hence $B(x; \varepsilon) \cap (X \setminus A)^- = \emptyset$, and so $B(x; \varepsilon) \cap (X \setminus A) = \emptyset$. Therefore $B(x; \varepsilon) \subset A$. Since $B(x; \varepsilon)$ is open and a subset of set $A$, then $x \in \text{int}(A)$. So $X \setminus (X \setminus A)^- \subset \text{int}(A)$. 

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Let \( x \in \text{int}(A) \). Then since \( \text{int}(A) \) is open (it is the union of a collection of open sets; use Theorem 1.9(c)). So (by definition) there is \( \varepsilon > 0 \) such that \( B(x; \varepsilon) \subset \text{int}(A) \). Now \( X \setminus B(x; \varepsilon) \) is closed and \( X \setminus A \subset X \setminus B(x; \varepsilon) \), so \( (X \setminus A)^- \subset X \setminus B(x; \varepsilon) \) (the set on the right hand side is closed by definition) and \( x \notin (X \setminus A)^- \). Therefore \( x \in X \setminus (X \setminus A)^- \) and so \( \text{int}(A) \subset X \setminus (X \setminus A)^- \). Hence, \( \text{int}(A) = X \setminus (X \setminus A)^- \). \( \square \)
Theorem II.1.13(f). Let $X$ be a metric space and $A \subset X$. Then $x_0 \in A^-$ if and only if for all $\epsilon > 0$, $B(x_0; \epsilon) \cap A \neq \emptyset$.

Proof. Let $x_0 \in A^- = X \setminus \text{int}(X \setminus A)$ by part (c2). Then $x_0 \not\in \text{int}(X \setminus A)$. 
Theorem II.1.13(f). Let $X$ be a metric space and $A \subset X$. Then $x_0 \in A^-$ if and only if for all $\epsilon > 0$, $B(x_0; \epsilon) \cap A \neq \emptyset$.

Proof. Let $x_0 \in A^- = X \setminus \text{int}(X \setminus A)$ by part (c2). Then $x_0 \notin \text{int}(X \setminus A)$. By part (e), for every $\epsilon > 0$ the ball $B(x_0; \epsilon)$ is not a subset of $X \setminus A$. That is, there is $y \in B(x_0; \epsilon) \cap A$ (for any $\epsilon > 0$ there is some such $y$).
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Now suppose $x_0 \notin A^- = X \setminus \text{int}(X \setminus A)$ by part (c2).
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Now suppose $x_0 \notin A^- = X \setminus \text{int}(X \setminus A)$ by part (c2). Then $x_0 \in \text{int}(X \setminus A)$ and, by part (e), there is $\varepsilon > 0$ such that $B(x_0; \varepsilon) \subset X \setminus A$. 


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Now suppose $x_0 \notin A^- = X \setminus \text{int}(X \setminus A)$ by part (c2). Then $x_0 \in \text{int}(X \setminus A)$ and, by part (e), there is $\varepsilon > 0$ such that $B(x_0; \varepsilon) \subset X \setminus A$. But then $B(x_0; \varepsilon) \cap A = \emptyset$ and $x_0$ does not satisfy the condition $B(x_0; \emptyset) \cap A \neq \emptyset$ (here, we have proven the contrapositive of the ‘only if’ part). \qed