Chapter II. Metric Spaces and the Topology of $\mathbb{C}$

II.1. Definitions and Examples of Metric Spaces—Proofs of Theorems
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Proof. Let $x \in X \setminus (X \setminus A)^-$. Well, $(X \setminus A)^-$ is closed (it is the intersection of a collection of closed sets; use Theorem 1.11(c)). So $X \setminus (X \setminus A)^-$ is open and so (by the definition of open) there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subset X \setminus (X \setminus A)^-$. 
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Theorem II.1.13(f). Let $X$ be a metric space and $A \subset X$. Then $x_0 \in A^-$ if and only if for all $\varepsilon > 0$, $B(x_0; \varepsilon) \cap A \neq \emptyset$.

Proof. Let $x_0 \in A^- = X \setminus \text{int}(X \setminus A)$ by part (c2). Then $x_0 \notin \text{int}(X \setminus A)$. By part (e), for every $\varepsilon > 0$ the ball $B(x_0; \varepsilon)$ is not a subset of $X \setminus A$. That is, there is $y \in B(x_0; \varepsilon) \cap A$ (for any $\varepsilon > 0$ there is some such $y$). Therefore, for all $\varepsilon > 0$, $B(x_0; \varepsilon) \cap A \neq \emptyset$. 
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