Note. Recall that a line in the complex plane can be represented by an equation of the form \( \text{Im}\left(\frac{z - a}{b}\right) = 0 \) where the line is “parallel” to the vector \( b \) and translated from the origin by an amount \( a \) (here we are knowingly blurring the distinction between vectors in \( \mathbb{R}^2 \) and numbers in \( \mathbb{C} \)).

We can represent a closed half plane with the equation \( \text{Im}\left(\frac{z - a}{b}\right) \leq 0 \). This represents the half plane to the right of the line \( \text{Im}\left(\frac{z - a}{b}\right) = 0 \) when traveling along the line in the “direction” of \( b \). This representation, along with some standard properties of logarithms and derivatives in the complex setting, allow us to prove the following so-called Gauss-Lucas Theorem (or sometimes simply the Lucas Theorem). For a reference, see page 29 of [1].

**Theorem 1. The Gauss-Lucas Theorem.**

*If all the zeros of a polynomial \( P(z) \) lie in a half plane in the complex plane, then all the zeros of the derivative \( P'(z) \) lie in the same half plane.*

**Proof.** By the Fundamental Theorem of Algebra, we can factor \( P \) as

\[ P(z) = a_n(z - r_1)(z - r_2) \cdots (z - r_n). \]
\[
\log P(z) = \log a_n + \log(z - r_1) + \log(z - r_2) + \cdots + \log(z - r_n)
\]

and differentiating both sides gives

\[
\frac{P'(z)}{P(z)} = \frac{1}{z - r_1} + \frac{1}{z - r_2} + \cdots + \frac{1}{z - r_n} = \sum_{k=1}^{n} \frac{1}{z - r_k}. (*)
\]

Suppose the half plane \(H\) that contains all the zeros of \(P(z)\) is described by \(\text{Im}((z - a)/b) \leq 0\). Then \(\text{Im}((r_k - a)/b) \leq 0\) for \(k = 1, 2, \ldots, n\). Now let \(z^*\) be some number not in \(H\). We want to show that \(P'(z^*) \neq 0\) (this will mean that all the zeros of \(P'(z)\) are in \(H\)). Well, \(\text{Im}((z^* - a)/b) > 0\). Let \(r_k\) be some zero of \(P\). Then

\[
\text{Im} \left( \frac{z^* - r_k}{b} \right) = \text{Im} \left( \frac{z^* - a - r_k + a}{b} \right) = \text{Im} \left( \frac{z^* - a}{b} \right) - \text{Im} \left( \frac{r_k - a}{b} \right) > 0.
\]

(Notice that \(\text{Im}((z^* - a)/b) > 0\) since \(z^*\) is not in \(H\), and \(-\text{Im}((r_k - a)/b) \geq 0\) since \(r_k\) is in \(H\).) The imaginary parts of reciprocal numbers have opposite signs, so \(\text{Im}(b/(z^* - r_k)) < 0\). Hence, by applying (*)

\[
\text{Im} \left( \frac{bP'(z^*)}{P(z^*)} \right) = \sum_{k=1}^{n} \text{Im} \left( \frac{b}{z^* - r_k} \right) < 0.
\]

So \(\frac{P'(z^*)}{P(z^*)} \neq 0\) and \(P'(z^*) \neq 0\). Therefore, if \(P'(z) = 0\), then \(z \in H\).

\[\blacksquare\]

**Note.** With repeated application of the Gauss-Lucas Theorem, we can prove the following corollary.

**Corollary 1.** The convex polygon in the complex plane which contains all the zeros of a polynomial \(P\), also contains all the zeros of \(P'\).
Note. For example, if \( P \) has eight zeros, then the convex polygon containing them might look like the following.

Note. There is no “clean” analogy of the Gauss-Lucas Theorem in the real setting (which is the case with many complex analysis results). For example, can you find a real polynomial with all of its zeros in a certain interval \([a, b]\), yet it has critical points outside of the interval? The answer: YES!

Note. A weaker version of Corollary 1 is the following.

Corollary 2. A circle which contains all of the zeros of polynomial \( P \), also contains all of the zeros of \( P' \).

Note. We now restrict our study to a class of polynomials which is sort of “normalized” with respect to the location of zeros. This is the set of polynomials each of which has all of its zeros in the unit disk of the complex plane, \( \{z \mid |z| \leq 1\} \). We can easily use Corollary 2 to get a result for this class of polynomials.
Corollary 3. If all the zeros of a polynomial $P$ lie in $|z| \leq 1$, then all the zeros of $P'$ also lie in $|z| \leq 1$.

Note. It is important that we are studying the set of all polynomials with their zeros in $|z| \leq 1$. We can violate Corollary 3 by considering a non-polynomial. Consider, for example, $f(z) = ze^{z/2}$. Then the only zero of $f$ is $z = 0$, and so all the zeros of $f$ lie in $|z| \leq 1$. However, $f'(z) = e^{z/2} + \frac{1}{2}ze^{z/2} = (\frac{1}{2}z + 1)e^{z/2}$. Then $f'$ has a zero at $z = -2$ and so there is a zero outside of $|z| \leq 1$.

Note. Another result concerning a relationship between the zeros of a polynomial and the zeros of the derivative involves the centroid of the zeros.

Definition. For polynomial $P(z) = a_n(z - r_1)(z - r_2) \cdots (z - r_n)$ with zeros $r_1, r_2, \ldots, r_n$ (not necessarily distinct), define the centroid of the zeros by placing a unit mass at each $r_k$ in the complex plane and then finding the center of mass of the resulting distribution (as in Calculus 2).

Theorem 2. The centroid of the zeros of a polynomial $P$ is the same as the centroid of the zeros of $P'$.

Proof. Theorem 2 is listed as an exercise in Morris Marden’s Geometry of Polynomials (see Exercise 4, page 53, [9]). Let the zeros of $n$-degree polynomial $P$ be $r_1, r_2, \ldots, r_n$, and so

$$P(z) = \sum_{k=0}^{n} a_k z^k = a_n \prod_{k=1}^{n} (z - r_k).$$

Then $P(z) = a_n z^n + a_n (-r_1 - r_2 - \cdots - r_n) z^{n-1} + \cdots + a_1 z + a_0$. So $a_{n-1} =$
\[-a_n(r_1 + r_2 + \cdots + r_n)\) and the centroid of the zeros of \(P\) is

\[
\frac{r_1 + r_2 + \cdots + r_n}{n} = \left(\frac{1}{n}\right) \left(\frac{-a_{n-1}}{a_n}\right) = \frac{-a_{n-1}}{na_n}.
\]

Let the zeros of \(P'\) be \(s_1, s_2, \ldots, s_{n-1}\). Then

\[
P'(z) = \sum_{k=1}^{n} ka_k z^{k-1} = na_n \prod_{k=1}^{n-1} (z - s_k).
\]

Similar to above, the centroid of the zeros of \(P'\) is

\[
\frac{s_1 + s_2 + \cdots + s_{n-1}}{n - 1} = \left(\frac{1}{n - 1}\right) \left(\frac{-(n - 1)a_{n-1}}{na_n}\right) = \frac{-a_{n-1}}{na_n}.
\]

Therefore, the centroid of the zeros of \(P\) and the centroid of the zeros of \(P'\) are the same.

**Note.** Now for the object of our interest! It is known variously as the Ilieff Conjecture, the Ilieff-Sendov Conjecture, and the Sendov Conjecture (making it particularly difficult to search for papers on the subject). It was originally posed by Bulgarian mathematician Blagovest Sendov in 1958 (according to [12]; sometimes the year 1962 is reported [11]), but often attributed (as Miller says in [11]) to Ilieff because of a reference in Hayman’s *Research Problems in Function Theory* in 1967 [8].

**Conjecture.** The Ilieff-Sendov Conjecture.

*If all the zeros of a polynomial \(P\) lie in \(|z| \leq 1\) and if \(r\) is a zero of \(P\), then there is a zero of \(P'\) in the circle \(|z - r| \leq 1\).*
Note. Combining the Ilieff-Sendov Conjecture with Corollary 3, we can further restrict the conjectured location of the critical points of $P$.

A Personal Note. I first saw a presentation on this conjecture in April 1986 at the “12th Annual Graduate Mathematics Conference” at Syracuse University. The presentation was by Michael Miller of LeMoyne College in Syracuse, NY. It was (in some part) this presentation that drew me away from discrete math and towards complex analysis for my Ph.D. topic.

Note. According to a recent (2008) paper by Michael Miller [12], there have been over 80 papers written on the conjecture. As a result, it has been demonstrated in many special cases. Some of the special cases are:

1. 3rd and 4th degree polynomials [13],
2. 5th degree polynomials [10],
3. polynomials having a root of modulus 1 [13, 15],
4. polynomials with real and non-positive coefficients [16],
5. polynomials with at most three distinct zeros [6, 14],
6. polynomials with at most six distinct zeros [3],
7. polynomials of degree less than or equal to 6 [4],

8. polynomials of degree less than or equal to 8 [5], and

9. the circle $|z - r| \leq 1.08331641$ [2].

**Note.** Many of the papers on the conjecture have been a bit more computational and have centered around finding “extremal” polynomials which push the locations of the zero of $P$ to the edge of the $|z - r| \leq 1$ region. This is especially true of the recent work of Micheal Miller [11, 12].

**Note.** A common approach to proving a difficult conjecture is to prove something even more restrictive than the conjecture, and then the conjecture falls as a corollary. This is how Andrew Wiles eventually gave a proof of Fermat’s Last Theorem—he proved the more general “Taniyama-Shimura Conjecture” for semistable elliptic curves, and from this Fermat’s Last Theorem followed.

In 1969, Goodman, Rahman, and Ratti [7] (and independently Schmiesser [15]) conjectured that the Ileff-Sendov Conjecture could be modified to the claim that (with the notation above) the region $|z - r/2| \leq 1 - |r|/2$ must contain a zero of $P'$. This is the blue region here:

![Diagram](image)

However, this conjecture is not true as shown by Micheal Miller in 1990 [11]. The following eighth degree polynomial violates the Goodman, Rahman,
Ratti Conjecture:

\[ P(z) = (z - 0.8)(z^7 + 1.241776468z^6 + 1.504033112z^5 + 1.702664563z^4 \]
\[ + 1.702664563z^3 + 1.504033112z^2 + 1.241776468z + 1) \].

Miller also found degree 6, 10, and 12 polynomials violating the new conjecture.

**Dr. Bob’s Conjecture.** In order to find a counterexample to the Ilieff-Sendov Conjecture, we would need to find a polynomial for which there is a zero of \( P \) at some point \( r \), yet the region (in yellow here) is free of zeros of \( P' \). This would force the zeros of \( P' \) to lie in the “lune-shaped” region (green here). However, if this is the case then the centroid of the zeros of \( P \) must lie rather far from \( r \). *It seems to me* that this could not happen in light of Theorem 2 and the location of \( r \). To my knowledge, no one has taken this approach...maybe with good reason!
Bibliography


*Last updated: September 11, 2017.*