I.5. Lines and Half-Planes in \( \mathbb{C} \)

**Recall.** The parametric equation for a line in \( \mathbb{R}^2 \) is \((x, y) = (a_1, a_2) + t(b_1, b_2)\) where \( t \in \mathbb{R} \). The line passes through the point \((a_1, a_2)\) (when \( t = 0 \)) and has direction vector \((b_1, b_2)\). So the equation of a line in \( \mathbb{C} \) is \( z = a + tb \) where \( t \in \mathbb{R} \) and \( a, b \in \mathbb{C} \). This can be rearranged as \( t = (z - a) / b \). Since \( t \) is real, the equation of a line in \( \mathbb{C} \) is of the form \( \text{Im}(\frac{z - a}{b}) = 0 \) where \( b \neq 0 \).

**Note.** Let \( a = 0 \) and \( b = \text{cis}(\beta) \) (i.e., without loss of generality \( |b| = 1 \)). Then the line is \( \text{Im}(z/b) = 0 \). If \( z = r\text{cis}(\theta) \) and \( \text{Im}(z/b) > 0 \), then \( \text{Im}(r\text{cis}(\theta - \beta)) = r\sin(\theta - \beta) > 0 \) and so \( r\sin(\theta - \beta) > 0 \). Now \( \sin(\theta - \beta) > 0 \) is satisfied when \( 0 < \theta - \beta < \pi \), or \( \beta < \theta < \pi + \beta \). So \( \text{Im}(z/b) > 0 \) is the half plane:
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If we translate all $z$ satisfying $\text{Im}(z/b) > 0$ by an amount $a$, we get the half plane:

Notice that if we interpret $b$ as a vector and we travel along the line in the direction $b$, then $\text{Im}\left(\frac{z-a}{b}\right) > 0$ lies to the left of the line and $\text{Im}\left(\frac{z-a}{b}\right) < 0$ lies to the right of the line.

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