Chapter III. Elementary Properties and Examples of Analytic Functions

III.1. Power Series

Note. We will see that classical complex analysis is about the study of functions with power series representations (Chapter III) and path integrals of such functions (Chapter IV).

Definition. The series \( \sum_{n=0}^{\infty} a_n \) converges to \( z \) if for all \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that for all \( m \geq N \), \( \left| \sum_{n=0}^{m} a_n - z \right| < \varepsilon \). That is, the sequence of partial sums \( s_m = \sum_{n=0}^{m} a_n \) converges to \( z \). The series \( \sum a_n \) converges absolutely if the series of real numbers \( \sum |a_n| \) converges.

Proposition III.1.1. If \( \sum a_n \) converges absolutely, then the series converges.

Idea of Proof. Follows from the Triangle Inequality and partial sums. \( \square \)

Definition. For a sequence \( \{a_n\} \subseteq \mathbb{R} \), define \( \liminf_{n \to \infty} a_n = \lim_{n \to \infty} (\inf\{a_n, a_{n+1}, \ldots\}) \) and \( \limsup_{n \to \infty} a_n = \lim_{n \to \infty} (\sup\{a_n, a_{n+1}, \ldots\}) \).
Note. The values of $\underline{\lim} a_n$ and $\overline{\lim} a_n$ could be $+\infty$ or $-\infty$ since these are defined in terms of suprema and infima. One shows in a Real Analysis class that (when finite) $\overline{\lim} a_n$ is the largest subsequential limit point for sequence $\{a_n\}$ and $\underline{\lim} a_n$ is the smallest subsequential limit point.

Note. The values $\underline{\lim} a_n$ and $\overline{\lim} a_n$ always exist and $\lim a_n$ exists if and only if $\underline{\lim} a_n = \overline{\lim} a_n$.

Example. Consider $\{a_n\} = \{\sin n\}$ ($n$ in radians). Then $\underline{\lim} a_n = -1$ and $\overline{\lim} a_n = 1$ (based on the transcendental-ness of $\pi$).

Question 1. Can you find a sequence with every natural number as a subsequential limit?

Answer. YES! Consider $\{1; 1, 2; 1, 2, 3; 1, 2, 3, 4; \ldots\}$.

Question 2. Can you find a sequence with every rational number as a subsequential limit?

Answer. YES! Let $\{a_n\}$ be an enumeration of the rationals and consider $\{q_1; q_1, q_2; q_1, q_2, q_3; q_1, q_2, q_3, q_4; \ldots\}$. 
Question 3. Can you find a sequence with every real number as a subsequential limit?

Answer. YES! Take the sequence \( \{q_n\} \) as above and use an \( \varepsilon \) argument. (Notice that for this sequence, \( \lim q_n = -\infty \) and \( \lim q_n = \infty \).)

Definition. A power series about \( a \in \mathbb{C} \) is an infinite series of the form \( \sum_{n=0}^{\infty} a_n(z - a)^n \). A geometric series is of the form \( \sum_{n=0}^{\infty} z^n \).

Note. As with geometric series of real numbers, \( \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z} \) if \( |z| < 1 \). If \( |z| \geq 1 \) then the series diverges to \( \infty \).

Theorem III.1.3. If \( \sum_{n=0}^{\infty} a_n(z - a)^n \), define the number \( R \) as \( \frac{1}{R} = \lim |a_n|^{1/n} \) (so \( 0 \leq R \leq \infty \)). Then

(a) if \( |z - a| < R \), the series converges absolutely,

(b) if \( |z - a| > R \), the series diverges, and

(c) if \( 0 < r < R \) then the series converges uniformly on \( |z - a| \leq r \). Moreover, \( R \) is the only number having properties (a) and (b). \( R \) is called the radius of convergence of the power series.
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Note. The following result gives a Ratio Test for complex power series.

**Proposition III.1.4.** If \( \sum_{n=0}^{\infty} a_n(z-a)^n \) is a given power series with radius of convergence \( R \), then \( R = \lim |a_n/a_{n+1}| \), if the limit exists.

Note. We can use power series to define functions in various regions of the complex plane (the region of convergence).

**Definition.** Define the exponential function \( e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \).

Note. By Proposition III.1.4, the radius of convergence of \( e^z \) is

\[
R = \lim \left| \frac{a_n}{a_{n+1}} \right| = \lim \left( \frac{1/n!}{1/(n+1)!} \right) = \infty.
\]

Note. The following deals with products and sums of series.

**Proposition III.1.6.** Let \( \sum a_n(z-a)^n \) and \( \sum b_n(z-a)^n \) be power series with radii of convergence \( \geq r > 0 \). Define \( c_k = \sum_{k=0} a_kb_{n-k} \). Then both power series \( \sum (a_n + b_n)(z-a)^n \) and \( \sum c_n(z-a)^n \) have radius of convergence \( \geq r \) and for \( |z-a| < r \):

\[
\sum (a_n + b_n)(z-a)^n = \sum a_n(z-a)^n + \sum b_n(z-a)^n \quad \text{and} \quad \sum c_n(z-a)^n = \left( \sum a_n(z-a)^n \right) \left( \sum b_n(z-a)^n \right).
\]

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