**IV.6. The Homotopic Version of Cauchy’s Theorem and Simple Connectivity**

**Note.** In this section, we give a more thorough version of Cauchy’s Theorem. Informally, we show that if path $\gamma_0$ can be continuously transformed into path $\gamma_1$ over the region of analyticity of function $f$, then $\int_{\gamma_0} f = \int_{\gamma_1} f$.

**Definition IV.6.1.** Let $\gamma_0, \gamma_1 : [0, 1] \to G$ be two closed rectifiable curves in a region $G$. Then $\gamma_0$ is *homotopic* to $\gamma_1$ in $G$ if there is a continuous function $\Gamma : [0, 1] \times [0, 1] \to G$ such that

\[
\begin{align*}
\Gamma(s, 0) &= \gamma_0(s) \text{ and } \Gamma(s, 1) = \gamma_1(s) \text{ for } s \in [0, 1] \\
\Gamma(0, t) &= \Gamma(1, t) \text{ for } t \in [0, 1].
\end{align*}
\]

This is denoted $\gamma_0 \sim \gamma_1$ (where $G$ is understood).

**Note.** The picture is:
Note. We have that for each \( t \in [0,1] \), \( \gamma_t(s) = \Gamma(s,t) \) is closed. However, we do not require \( \gamma_t(s) \) to be rectifiable, though this will, in practice, be the case.

Note. Homotopy \( \sim \) is an equivalence relation (see pages 88 and 89). In Section IV.4 we saw how to find the inverse of a path \( \gamma \) (we denoted the inverse as \( -\gamma \)) and how to add paths \( \gamma \) and \( \sigma \) where \( \gamma(1) = \sigma(0) \) (denoted \( \gamma + \sigma \); see page 81). We now give a brief description of the use of these ideas in the area of algebraic topology. This information can be found in Appendix A of Andrew Wallace’s Algebraic Topology: Homology and Cohomology (NY: W. A. Benjamin, 1970). This book is still in print through Dover Publications (Amazon.com’s “Look Inside” offers a complete view of Appendix A and Appendix B [accessed 3/30/2014]; the parenthetical references here are based on Wallace’s book). We can form a group out of the equivalence classes of closed paths “based” at some point \( x \in E \). It can also be shown that the set of equivalence classes (called homotopy classes) of closed paths based at \( x \) form a group under the operation + given in Section IV.4 (Theorem A-6). This group is called the fundamental group of \( E \) with respect to the base point \( x \) and is denoted \( \pi(E,x) \) (Definition A-9). If points \( x,y \in E \) can be joined by a path in \( E \), then \( \pi(E,x) \cong \pi(E,y) \) (Theorem A-7). In fact, the fundamental group of a space \( E \) is a topological invariant of the space; that is, if spaces \( E \) and \( F \) are homeomorphic (i.e., there is a one to one and onto continuous mapping from \( E \) to \( F \)) then the fundamental group of \( E \) is isomorphic to the fundamental group of \( F \) (Exercise A-4).
Definition. A set $G$ is convex if given any two points $a$ and $b$ in $G$, the line segment joining $a$ and $b$, $[a, b]$, lies entirely in $G$. The set $G$ is star shaped if there is a point $a$ in $G$ such that for each $z \in G$, the line segment $[a, z]$ lies entirely in $G$. Such a set is a-star shaped.

Proposition IV.6.4. Let $G$ be an open set which is a-star shaped. If $\gamma_0$ is the curve which is constantly equal to $a$ (that is, $\gamma_0(t) = a$ for $t \in [0, 1]$), then every closed rectifiable curve in $G$ is homotopic to $\gamma_0$.

Definition. If $\gamma$ is a closed rectifiable curve in $G$ then $\gamma$ is homotopic to zero ($\gamma \sim 0$) if $\gamma$ is homotopic to a constant curve.

Note. The equivalence class of all curves homotopic to zero form the identity element in the fundamental group of $G$ (hence the terminology).

Note. We now link Cauchy’s Theorem to curve homotopy. The Second Version of Cauchy’s Theorem is a special case of the Third Version. We offer a proof of the Third Version, so we skip a proof of the Second Version.
Theorem IV.6.6. Cauchy’s Theorem (Second Version).
If \( f : G \to \mathbb{C} \) is an analytic function and \( \gamma \) is a closed rectifiable curve in \( G \) such that \( \gamma \sim 0 \), then \( \int_{\gamma} f = 0 \).

Theorem IV.6.7. Cauchy’s Theorem (Third Version).
If \( \gamma_0 \) and \( \gamma_1 \) are two closed rectifiable curves in \( G \) and \( \gamma_0 \sim \gamma_1 \), then \( \int_{\gamma_0} f = \int_{\gamma_1} f \) for every function \( f \) analytic on \( G \).

Note. The BIG IDEA in the proof of Cauchy’s Theorem—Third Version is to get integrals over little quadrilaterals that lie inside small DISKS which are subsets of \( G \), then to use Proposition IV.2.15 (see page 73). Now for the lengthy proof.

Corollary IV.6.10. If \( \gamma \) is a closed rectifiable curve in \( G \) such that \( \gamma \sim 0 \), then \( n(\gamma; w) = 0 \) for all \( w \in \mathbb{C} \setminus G \).
Note. The converse of Corollary 6.10 does not hold. Consider, for example, Problem IV.6.8: Let $G = \mathbb{C} \setminus \{a, b\}$, $a \neq b$, and $\gamma$:

![Diagram](image1)

Then for all $w \in \mathbb{C} \setminus G$ (which is just $w = a$ and $w = b$) we have $n(\gamma; a) = n(\gamma; b) = 0$:

![Diagram](image2)

$n(\gamma; a) = 1 - 1 = 0$  
$n(\gamma; b) = 0 - 0 = 0$

However $\gamma \not\sim 0$ (“convince yourself” as the text says—imagine nails at $a$ and $b$ and $\gamma$ as a loop of string).
Note. We now consider non-closed rectifiable curves.

**Definition.** If \( \gamma_0, \gamma_1 : [0, 1] \to G \) are two rectifiable curves in \( G \) such that \( \gamma_0(0) = \gamma_1(0) = a \) and \( \gamma_0(1) = \gamma_1(1) = b \). Then \( \gamma_0 \) and \( \gamma_1 \) are fixed-end-point homotopic ("FEP" homotopic) if there is a continuous map \( \Gamma : [0, 1] \times [0, 1] \to G \) such that

\[
\begin{aligned}
\Gamma(s, 0) &= \gamma_0(s) \text{ and } \Gamma(s, 1) = \gamma_1(s) \text{ for } s \in [0, 1] \\
\Gamma(0, t) &= a \text{ and } \Gamma(1, t) = b \text{ for } t \in [0, 1].
\end{aligned}
\]

Note. If \( \gamma_0 \) and \( \gamma_1 \) are fixed-end-point homotopic, then \( \gamma_0 - \gamma_1 \sim 0 \) (see page 93 for details) and so by Cauchy’s Theorem—Second Version \( \int_{\gamma_0 - \gamma_1} f = 0 \) or \( \int_{\gamma_0} f = \int_{\gamma_1} f \) for all \( f \) analytic on \( G \) containing \( \gamma_0 \) and \( \gamma_1 \). That is:

**Theorem IV.6.13. Independence of Path Theorem.**

If \( \gamma_0 \) and \( \gamma_1 \) are two rectifiable curves in \( G \) from \( a \) and \( b \) and \( \gamma_0 \) and \( \gamma_1 \) are fixed-end-point homotopic then \( \int_{\gamma_0} f = \int_{\gamma_1} f \) for any function \( f \) analytic in \( G \).

Note. The following definition is really a topological concept.

**Definition.** An open set \( G \) is *simply connected* if \( G \) is connected and every closed curve \( G \) is homotopic to zero.
Note. A region is simply connected if it is connected and has “no holes”:

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{simply_connected}
\end{array}
\]

Simply Connected

Not Simply Connected

**Theorem IV.6.15. Cauchy’s Theorem (Fourth Version).**

If \( G \) is simply connected then \( \int_{\gamma} f = 0 \) for every closed rectifiable curve and every analytic function \( f \) on \( G \).

Note. The following brings primitives back into the picture (and recall that \( \int_{\gamma} f = 0 \) if \( \gamma \) is closed and \( f \) has a primitive).

**Corollary IV.6.16.** If open \( G \) is simply connected and \( f : G \to \mathbb{C} \) is analytic in \( G \) then \( f \) has a primitive in \( G \).

Note. The following is similar to Corollary 6.16, but deals with branches of the log.
Corollary IV.6.17. Let $G$ be simply connected and let $f : G \rightarrow \mathbb{C}$ be an analytic function such that $f(z) \neq 0$ for any $z \in G$. Then there is an analytic function $g : G \rightarrow \mathbb{C}$ such that $f(z) = \exp(g(z))$ (i.e., $g$ is a branch of $\log(f(z))$ on $G$). If $z_0 \in G$ and $e^{w_0} = f(z_0)$, we may choose $g$ such that $g(z_0) = w_0$.

Note. Corollary 6.17 verifies some observations we have made about branches of the log. For example, let $\sigma = \{z \mid z = te^{it}, t \in (0, \infty)\}$:

Then $G$ is simply connected, $f(z) = z$ is nonzero on $G$, and so there is a branch of the log on $G$. One such branch can be defined as follows. Consider a partition of $\mathbb{C} \setminus \sigma$ into regions:

For $z \in R_n$ define $\log(z) = \log|z| + \theta i$ where $\theta$ is an argument of $z$ in $(2\pi(n-1), 2\pi n]$.  

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