Titchmarsh’s “Number of Zeros in a Disk” Result

Jensen’s Formula. (From Conway’s *Function’s of One Complex Variable I*, page 280.) Let \( f \) be an analytic function on a region containing \( B(0; R) \) and suppose that \( a_1, a_2, \ldots, a_n \) are the zeros of \( f \) in \( B(0, R) \), repeated according to multiplicity. If \( f(0) \neq 0 \) then
\[
\log |f(0)| = - \sum_{k=1}^{n} \log \left| \frac{R}{a_k} \right| + \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(Re^{i\theta})| \, d\theta.
\]

Titchmarsh’s Number of Zeros Theorem. (Titchmarsh’s *The Theory of Functions*, page 171.) Let \( f \) be analytic in \( |z| < R \). Let \( |f(z)| \leq M \) in the disk \( |z| \leq R \) and suppose \( f(0) \neq 0 \). Then for \( 0 < \delta < 1 \) the number of zeros of \( f(z) \) in the disk \( |z| \leq \delta R \) is less than
\[
\frac{1}{\log 1/\delta} \log \frac{M}{|f(0)|}.
\]

**Proof.** Let \( f \) have \( n \) zeros in the disk \( |z| \leq \delta R \), say \( a_1, a_2, \ldots, a_n \). Then for \( 1 \leq k \leq n \) we have \( |a_k| \leq \delta R \), or \( \frac{R}{|a_k|} \geq \frac{1}{\delta} \). So
\[
\sum_{k=1}^{n} \log \frac{R}{|a_k|} = \log \frac{R}{|a_1|} + \log \frac{R}{|a_2|} + \cdots + \log \frac{R}{|a_n|} \geq n \log \frac{1}{\delta}.
\] (*)

By Jensen’s Formula, we have
\[
\sum_{k=1}^{n} \log \frac{R}{|a_k|} = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(Re^{i\theta})| \, d\theta - \log |f(0)|
\leq \frac{1}{2\pi} \int_{0}^{2\pi} \log M \, d\theta - \log |f(0)|
= \log M - \log |f(0)|
= \log \frac{M}{|f(0)|}.
\] (**)

Combining (*) and (**) gives
\[
n \log \frac{1}{\delta} \leq \sum_{k=1}^{n} \log \frac{R}{|a_k|} \leq \log \frac{M}{|f(0)|},
\]
or
\[
n \leq \frac{1}{\log 1/\delta} \log \frac{M}{|f(0)|}.
\]

Since \( n \) is the number of zeros of \( f \) in \( |z| \leq \delta R \), the result follows.