Chapter V. Singularities

V.1. Classification of Singularities

Note. In this section, we define various types of singularities of a function and develop the idea of a Laurent series.

Definition. A function $f$ has an isolated singularity at $z = a$ if there is an $R > 0$ such that $f$ is defined and analytic in $B(a; R) \setminus \{a\}$, but not in $B(a; R)$. Point $a$ is a removable singularity if there is an analytic function $g : B(a; R) \to \mathbb{C}$ such that $g(z) = f(z)$ for $0 < |z - a| < R$.

Example. Functions $f_1(z) = 1/z$ and $f_2(z) = \sin z/z$, and $f_3(z) = \exp(1/z)$ each have isolated singularities at $z = 0$. As shown in Exercise V.1.1, $f_2(z) = \sin z/z$ has a removable singularity.

Theorem V.1.2. If $f$ has an isolated singularity at $a$ then $z = a$ is a removable singularity if and only if $\lim_{z \to a} (z - a)f(z) = 0$.

Definition. If $z = a$ is an isolated singularity of $f$ then $a$ is a pole of $f$ if $\lim_{z \to a} |f(z)| = \infty$. If an isolated singularity is neither a pole nor a removable singularity it is called an essential singularity.
**Example.** Function $f(z) = \frac{1}{(z-a)^m}$ for $m \in \mathbb{N}$ has a pole at $z = a$. Function $g(z) = \exp(z^{-1})$ has an essential singularity at $z = 0$. In fact, a function with a pole at $z = a$ has a well defined form, as given next.

**Proposition V.1.4.** If $G$ is a region with $a \in G$, and if $f$ is analytic in $G \setminus \{a\}$ with a pole at $z = a$, then there is a positive integer $m$ and an analytic function $g : G \rightarrow \mathbb{C}$ such that $f(z) = \frac{g(z)}{(z-a)^m}$.

**Definition.** If $f$ has a pole at $z = a$ and $m$ is the smallest positive integer such that $f(z)(z-a)^m$ has a removable singularity at $z = a$, then $f$ has a pole of order $m$ at $z = a$. A pole of order 1 is called a simple pole.

**Note.** If $f$ has a pole of order $m$ at $z = a$, then $f(z) = g(z)/(z-a)^m$ where $g$ is analytic in $B(a; R)$ (for some $R > 0$), so

$$g(z) = A_m + A_{m-1}(z-a) + \cdots + A_1(z-a)^{m-1} + (z-a)^m \sum_{k=0}^{\infty} a_k(z-a)^k$$

and

$$f(z) = \frac{A_m}{(z-a)^m} + \frac{A_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{A_1}{(z-a)} + g_1(z)$$

where $g_1$ is analytic in $B(a; R)$ and $A_m \neq 0$. 
**Definition.** If \( f \) has a pole of order \( m \) at \( z = a \) and \( f \) satisfies

\[
f(z) = \frac{A_m}{(z-a)^m} + \frac{A_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{A_1}{(z-a)} + g_1(z)
\]

then

\[
\frac{A_m}{(z-a)^m} + \frac{A_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{A_1}{(z-a)}
\]

is called the *singular part* of \( f \) at \( z = a \).

**Note.** We will see that an essential singularity behaves rather like a pole of infinite order. This then produces an infinite singular part. First, some definitions.

**Definition V.1.10.** If \( \{z_n \mid n \in \mathbb{Z}\} \) is a doubly infinite sequence of complex numbers, then \( \sum_{n=-\infty}^{\infty} a_n \) is *absolutely convergent* if both \( \sum_{n=0}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} a_{-n} \) are absolutely convergent. If these series are absolutely convergent then define

\[
\sum_{n=-\infty}^{\infty} z_n = \sum_{n=1}^{\infty} z_{-n} + \sum_{n=0}^{\infty} z_n.
\]

If \( u_n \) is a function on a set \( S \) for \( n \in \mathbb{Z} \) and \( \sum_{n=-\infty}^{\infty} u_n(s) \) is absolutely convergent for every \( s \in S \), then the convergence is *uniform* over \( S \) if both \( \sum_{n=0}^{\infty} u_n \) and \( \sum_{n=1}^{\infty} u_{-n} \) converge uniformly on \( S \).

**Definition.** If \( 0 \leq R_1 < R_2 \leq \infty \) and \( a \) is any complex number, define

\[
\text{ann}(a; R_1, R_2) = \{ z \mid R_1 < |z-a| < R_2 \}.
\]
Note. We now deal with a series representation of a function analytic on an annulus.

Theorem V.1.11. Laurent Series Development.

Let \( f \) be analytic in \( \text{ann}(a; R_1, R_2) \). Then
\[
f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n
\]
where the convergence is absolute and uniform over the closure of \( \text{ann}(a; r_1, r_2) \) if \( R_1 < r_1 < r_2 < R_2 \). The coefficients \( a_n \) are given by
\[
a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{n+1}}\,dz \quad (1.12)
\]
where \( \gamma \) is the circle \(|z-a|=r\) for any \( r \) with \( R_1 < r < R_2 \). Moreover, this series is unique.

Note. The proof of Theorem V.1.11 is in a, sort of, self contained supplement.

The Laurent series allows us to classify isolated singularities.

Corollary V.1.18. Let \( z = a \) be an isolated singularity of \( f \) and let \( f(z) = \sum_{n=-\infty}^{\infty} a_n (a-z)^n \) be its Laurent expansion in \( \text{ann}(a;0,R) \). Then
(a) \( z = a \) is a removable singularity if and only if \( a_n = 0 \) for \( n \leq -1 \),
(b) \( a = z \) is a pole of order \( m \) if and only if \( a_{-m} \neq 0 \) and \( a_n = 0 \) for \( n \leq -(m+1) \), and
(c) \( z = a \) is an essential singularity if and only if \( a_n \neq 0 \) for infinitely many negative integers \( n \).
Note. If $f$ has an essential singularity at $z = a$, then $\lim_{z \to a} |f(z)|$ does not exist.

The text says: “This means that as $z$ approaches $a$ the values of $f(z)$ must wander through $\mathbb{C}$.” The following result shows that this wandering is very intense.

**Theorem V.1.21. Casorati-Weierstrass Theorem.**

If $f$ has an essential singularity at $z = a$ then for every $\delta > 0$, $\{f(\text{ann}(a; 0, \delta))\}^- = \mathbb{C}$.

Note. A more general result concerning the behavior of $f$ near an essential singularity is in Chapter XII (the last chapter of the text, page 300):

**Great Picard Theorem.**

Suppose an analytic function has an essential singularity at $z = a$. Then in each neighborhood of $a$, $f$ assumes each complex number, with one possible exception, an infinite number of times.

Note. Function $f(z) = \exp(1/z)$ is such a function, and it clearly does not take on the value 0.

Note. For the record, from page 297 we have:

**Little Picard Theorem.**

If $f$ is an entire function that omits two values, then $f$ is a constant.

Note. Of course, $f(z) = e^z$ is an example of a function omitting one value.