Chapter V. Singularities

V.1. Classification of Singularities

Note. In this section, we define various types of singularities of a function and develop the idea of a Laurent series.

Definition. A function $f$ has an isolated singularity at $z = a$ if there is an $R > 0$ such that $f$ is defined and analytic in $B(a; R) \setminus \{a\}$, but not in $B(a; R)$. Point $a$ is a removable singularity if there is an analytic function $g : B(a; R) \to \mathbb{C}$ such that $g(z) = f(z)$ for $0 < |z - a| < R$.

Example. Functions $f_1(z) = 1/z$ and $f_2(z) = \sin z/z$, and $f_3(z) = \exp(1/z)$ each have isolated singularities at $z = 0$. As shown in Exercise V.1.1, $f_2(z) = \sin z/z$ has a removable singularity.

Theorem V.1.2. If $f$ has an isolated singularity at $a$ then $z = a$ is a removable singularity if and only if $\lim_{z \to a} (z - a)f(z) = 0$.

Definition. If $z = a$ is an isolated singularity of $f$ then $a$ is a pole of $f$ if $\lim_{z \to a} |f(z)| = \infty$. If an isolated singularity is neither a pole nor a removable singularity it is called an essential singularity.
Example. Function \( f(z) = \frac{1}{(z-a)^m} \) for \( m \in \mathbb{N} \) has a pole at \( z = a \). Function \( g(z) = \exp(z^{-1}) \) has an essential singularity at \( z = 0 \). In fact, a function with a pole at \( z = a \) has a well defined form, as given next.

**Proposition V.1.4.** If \( G \) is a region with \( a \in G \), and if \( f \) is analytic in \( G \setminus \{a\} \) with a pole at \( z = a \), then there is a positive integer \( m \) and an analytic function \( g : G \to \mathbb{C} \) such that \( f(z) = \frac{g(z)}{(z-a)^m} \).

**Definition.** If \( f \) has a pole at \( z = a \) and \( m \) is the smallest positive integer such that \( f(z)(z-a)^m \) has a removable singularity at \( z = a \), then \( f \) has a **pole of order** \( m \) at \( z = a \). A pole of order 1 is called a **simple pole**.

**Note.** If \( f \) has a pole of order \( m \) at \( z = a \), then \( f(z) = g(z)/(z-a)^m \) where \( g \) is analytic in \( B(a;R) \) (for some \( R > 0 \)), so

\[
g(z) = A_m + A_{m-1}(z-a) + \cdots + A_1(z-a)^{m-1} + (z-a)^m \sum_{k=0}^{\infty} a_k(z-a)^k
\]

and

\[
f(z) = \frac{A_m}{(z-a)^m} + \frac{A_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{A_1}{(z-a)} + g_1(z)
\]

where \( g_1 \) is analytic in \( B(a;R) \) and \( A_m \neq 0 \).
**Definition.** If $f$ has a pole of order $m$ at $z = a$ and $f$ satisfies

$$f(z) = \frac{A_m}{(z-a)^m} + \frac{A_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{A_1}{(z-a)} + g_1(z)$$

then

$$\frac{A_m}{(z-a)^m} + \frac{A_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{A_1}{(z-a)}$$

is called the *singular part* of $f$ at $z = a$.

**Note.** We will see that an essential singularity behaves rather like a pole of infinite order. This then produces an infinite singular part. First, some definitions.

**Definition V.1.10.** If $\{z_n \mid n \in \mathbb{Z}\}$ is a doubly infinite sequence of complex numbers, then $\sum_{n=-\infty}^{\infty} a_n$ is *absolutely convergent* if both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{-n}$ are absolutely convergent. If these series are absolutely convergent then define

$$\sum_{n=-\infty}^{\infty} z_n = \sum_{n=1}^{\infty} z_{-n} + \sum_{n=0}^{\infty} z_n.$$

If $u_n$ is a function on a set $S$ for $n \in \mathbb{Z}$ and $\sum_{n=-\infty}^{\infty} u_n(s)$ is absolutely convergent for every $s \in S$, then the convergence is *uniform* over $S$ if both $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=1}^{\infty} u_{-n}$ converge uniformly on $S$.

**Definition.** If $0 \leq R_1 < R_2 \leq \infty$ and $a$ is any complex number, define

$$\text{ann}(a; R_1, R_2) = \{ z \mid R_1 < |z-a| < R_2 \}.$$
Note. We now deal with a series representation of a function analytic on an annulus.

Theorem V.1.11. Laurent Series Development.

Let $f$ be analytic in $\text{ann}(a; R_1, R_2)$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n$$

where the convergence is absolute and uniform over the closure of $\text{ann}(a; r_1, r_2)$ if $R_1 < r_1 < r_2 < R_2$. The coefficients $a_n$ are given by

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{n+1}} \,dz \quad (1.12)$$

where $\gamma$ is the circle $|z-a| = r$ for any $r$ with $R_1 < r < R_2$. Moreover, this series is unique.

Note. The proof of Theorem V.1.11 is in a, sort of, self contained supplement.

The Laurent series allows us to classify isolated singularities.

Corollary V.1.18. Let $z = a$ be an isolated singularity of $f$ and let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n$ be its Laurent expansion in $\text{ann}(a; 0, R)$. Then

(a) $z = a$ is a removable singularity if and only if $a_n = 0$ for $n \leq -1$,

(b) $a = z$ is a pole of order $m$ if and only if $a_{-m} \neq 0$ and $a_n = 0$ for $n \leq -(m+1)$, and

(c) $z = a$ is an essential singularity if and only if $a_n \neq 0$ for infinitely many negative integers $n$. 
Note. If \( f \) has an essential singularity at \( z = a \), then \( \lim_{z \to a} |f(z)| \) does not exist. The text says: “This means that as \( z \) approaches \( a \) the values of \( f(z) \) must wander through \( \mathbb{C} \).” The following result shows that this wandering is very intense.

**Theorem V.1.21. Casorati-Weierstrass Theorem.**

If \( f \) has an essential singularity at \( z = a \) then for every \( \delta > 0 \), \( \{ f(\text{ann}(a; 0, \delta)) \}^- = \mathbb{C} \).

Note. A more general result concerning the behavior of \( f \) near an essential singularity is in Chapter XII (the last chapter of the text, page 300):

**Great Picard Theorem.**

Suppose an analytic function has an essential singularity at \( z = a \). Then in each neighborhood of \( a \), \( f \) assumes each complex number, with one possible exception, an infinite number of times.

Note. Function \( f(z) = \exp(1/z) \) is such a function, and it clearly does not take on the value 0.

Note. For the record, from page 297 we have:

**Little Picard Theorem.**

If \( f \) is an entire function that omits two values, then \( f \) is a constant.

Note. Of course, \( f(z) = e^z \) is an example of a function omitting one value.

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