V.2. Residues

Note. Given our previous experience with integrals over closed and rectifiable
curves, we expect lots of integrals to be 0, except those related to $1/(z - a)$. Hence,
in a Laurent expansion, our attention is drawn to $a_{-1}$. In this section, we also
develop some techniques with which we can evaluate integrals of functions of a real
variable.

Definition V.2.1. Let $f$ have an isolated singularity at $z = a$ and let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - a)^n$ be its Laurent expansion about $z = a$. The residue of $f$ at $z = a$
is the coefficient $a_{-1}$, denoted $\text{Res}(f; a) = a_{-1}$.

Note. The following relates residues to integrals and winding numbers.

Theorem V.2.2. Residue Theorem.

Let $f$ be analytic in the region $G$, except for the isolated singularities $a_1, a_2, \ldots a_m$.
If $\gamma$ is a closed rectifiable curve in $G$ which does not pass through any of the points
$a_k$ and if $\gamma \approx 0$ in $G$ then

$$\frac{1}{2\pi i} \int_{\gamma} f = \sum_{k=1}^{m} n(\gamma; a_k) \text{Res}(f; a_k).$$
Note. The Residue Theorem allows us to evaluate certain integrals, provided we can evaluate winding numbers and residues. The following result allows us to compute residues in terms of derivatives.

**Proposition V.2.4.** Suppose \( f \) has a pole of order \( m \) at \( z = a \). Let \( g(z) = (z - a)^m f(z) \). Then

\[
\text{Res}(f; a) = \frac{1}{(m-1)!} g^{(m-1)}(a).
\]

**Note V.2.A.** If \( z = a \) is a simple pole of \( f \), then

\[
f(z) = \frac{a_{-1}}{z - a} + \sum_{k=0}^{\infty} a_k (z - a)^k \quad \text{and} \quad \text{Res}(f; a) = \lim_{z \to a} (z - a) f(z) = a_{-1}.
\]

**Example V.2.5.** Show

\[
\int_{-\infty}^{\infty} \frac{x^2}{1 + x^4} \, dx = \frac{\pi}{\sqrt{2}}.
\]

**Solution.** With \( f(z) = z^2/(1 + z^4) \), \( f \) has simple poles at the 4th roots of \(-1\), \( a_1 = \exp(\pi i/4) \), \( a_2 = \exp(3\pi i/4) \), \( a_3 = \exp(5\pi i/4) \), and \( a_4 = \exp(7\pi i/4) \). So by Note V.2.A,

\[
\text{Res}(f; a_1) = \lim_{z \to a_1} (z - a_1) f(z) = \lim_{z \to a_1} (z - a_1) \frac{z^2}{(z - a_1)(z - a_2)(z - a_3)(z - a_4)} = \frac{a_1^2}{((a_1 - a_2)(a_1 - a_3)(a_1 - a_4)} = \frac{i}{(2/\sqrt{2})(2/\sqrt{2}+2i/\sqrt{2})(2i/\sqrt{2})} = \frac{2\sqrt{2}}{4i(2+2i)} \left( \frac{2-2i}{2-2i} \right) = \frac{4\sqrt{2}(1-i)}{4i8} = \frac{1-i}{4\sqrt{2}},
\]

and

\[
\text{Res}(f; a_2) = \lim_{z \to a_2} (z - a_2) \frac{z^2}{(z - a_1)(z - a_2)(z - a_3)(z - a_4)}
\]
\[
\frac{a_2}{(a_2 - a_1)(a_2 - a_3)(a_2 - a_4)} = \frac{-i}{(-2/\sqrt{2})(2i/\sqrt{2})(-2/\sqrt{2} + 2i/\sqrt{2})}
\]
\[
= \frac{-2\sqrt{2}i}{(-2)(2i)(-2 + 2i)} = \frac{\sqrt{2}}{-4(1-i)} \left( \frac{1+i}{1+i} \right) = \sqrt{2}(1+i) - 8 = \frac{-1-i}{4\sqrt{2}}.
\]

Let \( R > 1 \) and let \( \gamma \) be the closed path:

Then by the Residue Theorem (Theorem V.2.2),
\[
\frac{1}{2\pi i} \int_{\gamma} f(z) \, dz = \text{Res}(f; a_1) + \text{Res}(f; a_2) = \frac{1-i}{4\sqrt{2}} + \frac{-1-i}{4\sqrt{2}} = \frac{-i}{2\sqrt{2}}.
\]

But breaking \( \gamma \) into the interval \([-R, R] \subset \mathbb{R} \) and the semicircle \( \{Re^{it} \mid 0 \leq t \leq \pi\} \) gives
\[
\frac{1}{2\pi i} \int_{\gamma} f(z) \, dz = \frac{1}{2\pi i} \int_{-R}^{R} \frac{x^2}{1+x^4} \, dx + \frac{1}{2\pi i} \int_{0}^{\pi} \frac{R^2 e^{2it}}{1 + R^4 e^{4it}} iRe^{it} \, dt
\]
and so
\[
\int_{-R}^{R} \frac{x^2}{1+x^4} \, dx = \frac{\pi}{\sqrt{2}} - iR^3 \int_{0}^{\pi} \frac{e^{3it}}{1 + R^4 e^{4it}} \, dt.
\]
For \( 0 \leq t \leq \pi \), \( |1 + R^4 e^{4it}| \geq R^4 - 1 \), so
\[
\left| iR^3 \int_{0}^{\pi} \frac{3e^{it}}{1 + R^4 e^{4it}} \, dt \right| \leq \frac{\pi R^3}{R^4 - 1}.
\]
So
\[
\lim_{R \to \infty} \left( iR^3 \int_{0}^{\pi} \frac{e^{3it}}{1 + R^4 e^{4it}} \, dt \right) = 0
\]
and
\[
\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} \, dx = \lim_{R \to \infty} \left( \int_{-R}^{R} \frac{x^2}{1+x^4} \, dx \right) = \lim_{R \to \infty} \left( \frac{\pi}{\sqrt{2}} - i R^3 \int_0^\pi \frac{e^{3it}}{1+R^4 e^{4it}} \, dt \right) = \frac{\pi}{\sqrt{2}}.
\]
Notice that this was evaluated using residues and no mention is made of antiderivatives!

**Example V.2.12.** Show that
\[
\int_0^\infty \frac{x^{-c}}{1+x} \, dx = \frac{\pi}{\sin(\pi c)} \text{ for } 0 < c < 1.
\]

**Solution.** We want to use residues and circles around 0 to set up a contour integral that produces the desired real definite integral in the limit. However, \(z^{-c} = \exp(-c \log z)\) requires a branch of the logarithm and so a branch cut from 0 to \(\infty\).

Let \(G = \{z \mid z \neq 0 \text{ and } 0 < \arg(z) < 2\pi\}\). Define a branch of the logarithm on \(G\) of \(\ell(z) = \ell(re^{i\theta}) = \log(r) + i\theta\) where \(0 < \theta \leq 2\pi\). Then on \(G\), \(f(z) = \exp(-c\ell(z))\) is a branch of \(z^{-c}\). Now we define contour \(\gamma\) over which we will integrate. Let \(0 < r < a < R\) and let \(\delta > 0\). Let \(L_1\) be the line segment \(r + \delta i, R + \delta i\), let \(\gamma_R\) be the part of the circle \(|z| = R\) from \(R + \delta i\) counterclockwise to \(R - \delta i\), let \(L_2\) be the line segment \([R - \delta i, r - \delta i]\), and let \(\gamma_r\) be the part of the circle \(|z| = r\) from \(r - \delta i\) clockwise to \(r + \delta i\). Put \(\gamma = L_1 + \gamma_R + L_2 + \gamma_r\). See figure the figure below.
Then $\{\gamma\} \subset G$, $\gamma \sim 0$ in $G$, and $-1$ is inside $\gamma$. Now $z^{-c}/(1 + z)$ has a simple pole at $z = -1$ so by Note V.2.A,

$$\text{Res}(z^{-c}/(1 + z); -1) = \lim_{z \to -1} (1 + z)(z^{-c}/(1 + z)) = \lim_{z \to -1} z^{-c} = f(-1) = \exp(-c(\log(1) + i(\pi))) = e^{-ci\pi}.$$ 

By the Residue Theorem (Theorem V.2.2),

$$\int_{\gamma} \frac{z^{-c}}{1 + z} \, dz = 2\pi i \text{Res}(z^{-c}/(1 + z)) = 2\pi i e^{-ci\pi}.$$

Now $L_1 = [r + \delta i, R + \delta i]$ can be parameterized as $L_1(t) = t + \delta i$ for $t \in [r, R]$. Then with $f(z) = z^{-c}$,

$$\int_{L_1} \frac{f(z)}{1 + z} \, dz = \int_r^R \frac{f(t + i\delta)}{1 + t + i\delta} \, dt.$$

To deal with this integral, we define $g(t, \delta)$ on compact set $[r, R] \times [0, \pi/2]$ as

$$g(t, \delta) = \begin{cases} \left| \frac{f(t+i\delta)}{1+t+i\delta} - \frac{t^{-c}}{1+t} \right| & \text{if } \delta \in (0, \pi/2] \\ 0 & \text{if } \delta = 0. \end{cases}$$

Then $g$ is continuous and so, by Theorem II.5.15, uniformly continuous. So if $\varepsilon > 0$ then there is $\delta_0 > 0$ such that if $(t-t')^2 + (\delta - \delta')^2 < \delta_0^2$ then $|g(t, \delta) - g(t', \delta')| < \varepsilon/R$. In particular, with $t = t'$ and $\delta' = 0$, we have $g(t', \delta') = g(t', 0) = 0$ and so for $(t-t')^2 + (\delta - \delta') = \delta_0^2$ (or $\delta < \delta_0$), $|g(t, \delta) - g(t', \delta')| = g(t, \delta) < \varepsilon/R$. So for $\delta < \delta_0$ we have $\int_r^R g(t, \delta) \, dt \leq (\varepsilon/R)R = \varepsilon$. So $\lim_{\delta \to 0^+} \int_r^R g(t, \delta) \, dt = 0$ and

$$\lim_{\delta \to 0^+} \left| \int_r^R \frac{f(t+i\delta)}{1+t+i\delta} \, dt - \int_r^R \frac{t^{-c}}{1+t} \, dt \right| = \lim_{\delta \to 0^+} \left| \int_r^R \left( \frac{f(t+i\delta)}{1+t+i\delta} - \frac{t^{-c}}{1+t} \right) \, dt \right|$$

$$\leq \lim_{\delta \to 0^+} \left( \int_r^R \left| \frac{f(t+i\delta)}{1+t+i\delta} - \frac{t^{-c}}{1+t} \right| \, dt \right) = \lim_{\delta \to 0^+} \left( \int_r^R g(t, \delta) \, dt \right) = 0,$$
and so
\[
\lim_{\delta \to 0^+} \left( \int_r^R \frac{f(t + i\delta)}{1 + t + i\delta} \, dt \right) = \int_r^R \frac{t^{-c}}{1 + t} \, dt \quad \text{or} \quad \lim_{\delta \to 0^+} \left( \int_r^R \frac{f(z)}{1 + z} \, dz \right) = \int_r^R \frac{t^{-c}}{1 + t} \, dt.
\]

Similarly, as is to be shown in Exercise V.2.A,
\[
-e^{-2\pi i} \int_r^R \frac{t^{-c}}{1 + t} \, dt = \lim_{\delta \to 0^+} \int_{L_2} \frac{f(z)}{1 + z} \, dz.
\]

As shown above, \( \int_{\gamma} \frac{f(z)}{1 + z} \, dz = 2\pi i e^{-\pi c} \) and this is independent of \( \delta \) so letting \( \delta \to 0^+ \) we have
\[
2\pi i e^{-\pi c} = \lim_{\delta \to 0^+} \left( \int_{\gamma} \frac{f(z)}{1 + z} \, dz \right) = \lim_{\delta \to 0^+} \left( \int_{L_1} \frac{f(z)}{1 + z} \, dz + \int_{\gamma_R} \frac{f(z)}{1 + z} \, dz \right.
\]
\[
+ \int_{L_2} \frac{f(z)}{1 + z} \, dz + \int_{\gamma_r} \frac{f(z)}{1 + z} \, dz \Bigg) = \int_r^R \frac{t^{-c}}{1 + t} \, dt - e^{-i\pi c} \int_r^R \frac{t^{-c}}{1 + t} \, dt + \lim_{\delta \to 0^+} \left( \int_{\gamma_r} \frac{f(z)}{1 + z} \, dz + \int_{\gamma_R} \frac{f(z)}{1 + z} \, dz \right). \quad (2.16)
\]

Now if \( \rho > 0 \) and \( \rho \neq 1 \) and if \( \gamma_\rho \) is the part of the circle \( |z| = \rho \) from \( \sqrt{\rho^2 - \delta^2} + i\delta \) to \( \sqrt{\rho^2 - \delta^2} - i\delta \) then
\[
\left| \int_{\gamma_\rho} \frac{f(z)}{1 + z} \, dz \right| \leq \int_{\gamma_\rho} \left| \frac{f(z)}{1 + z} \right| \, |dz| \leq \frac{\rho^{-c}}{|1 - \rho|} 2\pi \rho.
\]

Since this bound is independent of \( \delta \), from (2.16), we have
\[
\left| 2\pi i e^{-i\pi c} - (1 - e^{-i\pi c}) \int_r^R \frac{t^{-c}}{1 + t} \, dt \right| \leq \frac{r^{-c}}{|1 - r|} 2\pi r + \frac{R^{-c}}{|1 - R|} 2\pi R.
\]

Now taking limits \( r \to 0^+ \) and \( R \to \infty \), \( \frac{r^{-c}}{|1 - r|} 2\pi r \to 0 \) and \( \frac{R^{-c}}{|1 - R|} 2\pi R \to 0 \) since \( 0 < c < 1 \). Hence
\[
(1 - e^{-2\pi ic}) \int_0^\infty \frac{t^{-c}}{1 + t} \, dt = 2\pi i e^{-i\pi c}
\]
or

\[ \int_0^\infty \frac{t^{-c}}{1 + t} \, dt = \frac{2\pi i e^{-i\pi c}}{1 - e^{-2i\pi c}} = \frac{2\pi i}{e^{i\pi c} - e^{-i\pi c}} = \frac{2\pi i}{2i \sin(\pi c)} = \frac{\pi}{\sin(\pi c)} \]

since \( \sin z = \frac{(e^{iz} - e^{-iz})}{(2i)} \).

**Note.** A much easier solution to the previous example is given in *Schaum’s Outline Series, Complex Variables* by Murray Spiegel [1964], page 185. Unfortunately, it is incorrect! Effectively, Spiegel takes \( \delta = 0 \) in our notation. But then \( \gamma_r \) and \( \gamma_R \) both must contain points on the branch cut of the logarithm (and hence on the branch cut of \( z^{-c} \)).

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