V.3. The Argument Principle

**Note.** In this section, we concentrate on zeros and poles of a function. In the Argument Principle we relate the value of an integral to winding numbers of zeros or poles. In Rouche’s Theorem, a quantity related to the number of zeros and the number of poles is given which is preserved between functions satisfying a certain (inequality) relationship. The specific class of functions of concern is defined in the following.

**Definition V.3.3.** If $G$ is open and $f$ is a function defined and analytic on $G$ except for poles, then $f$ is a meromorphic function on $G$.

**Note.** If $f$ is meromorphic on $G$, then we can define $f : G \to \mathbb{C}_\infty$ by setting $f(z) = \infty$ at each pole of $f$. By Exercise V.3.4 $f$ is then a continuous mapping where we treat $\mathbb{C}_\infty$ as a metric space with the metric given in section I.6.

**Note.** If $f$ is analytic at $z = a$ and $f$ has a zero of order $m$ at $z = a$, then $f(z) = (z - a)^m g(z)$ where $g(a) \neq 0$ be Definition IV.3.1. Hence

$$
\frac{f'(z)}{f(z)} = \frac{m(z - a)^{m-1}g(z) + (z - a)^m g'(z)}{(z - a)^m g(z)} = \frac{m}{z - a} + \frac{g'(z)}{g(z)}. \quad (3.1)
$$

Since $g(a) \neq 0$, then $g'/g$ is analytic “near” $z = a$. 
Note. If \( f \) has a pole of order \( m \) at \( z = a \), then \( f(z) = (z - a)^{-m}g(z) \) where \( g \) is analytic at \( z = a \) and \( g(a) \neq 0 \) by the definition of pole of order \( m \) and Proposition 1.6. Then

\[
\frac{f'(z)}{f(z)} = \frac{-m(z - a)^{-m-1}g(z) + (z - a)^{-m}g'(z)}{(z - a)^{-m}g(z)} = \frac{-m}{z - a} + \frac{g'(z)}{g(z)}. \tag{3.2}
\]

Again, since \( g(a) \neq 0 \), then \( g'/g \) is analytic “near” \( z = a \).

**Theorem V.3.4. Argument Principle.**

Let \( f \) be meromorphic in \( G \) with poles \( p_1, p_2, \ldots, p_m \) and zeros \( z_1, z_2, \ldots, z_n \) repeated according to multiplicity. If \( \gamma \) is a closed rectifiable curve in \( G \) where \( \gamma \approx 0 \) and not passing through \( p_1, p_2, \ldots, p_m, z_1, z_2, \ldots, z_n \), then

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum_{k=1}^{n} n(\gamma; z_k) - \sum_{j=1}^{m} n(\gamma; p_j).
\]

**Note.** Given the representation of \( f'/f \) given in the proof, we see that winding numbers naturally arise here. Also, we would expect a primitive of \( f'/f \) to be \( \log(f) \), which of course does not exist on \{\gamma\} (unless the winding numbers are 0), but again this hints at multiples of \( 2\pi i \).

**Theorem V.3.6.** Let \( f \) be meromorphic on region \( G \) with zeros \( z_1, z_2, \ldots, z_n \) and poles \( p_1, p_2, \ldots, p_m \) repeated according to multiplicity. If \( g \) is analytic on \( G \) and \( \gamma \) is a closed rectifiable curve in \( G \) where \( \gamma \approx 0 \) and \( \gamma \) does not pass through any zero or pole of \( f \), then

\[
\frac{1}{2\pi i} \int_{\gamma} g(z) \frac{f'(z)}{f(z)} \, dz = \sum_{k=1}^{n} g(z_k)n(\gamma; z_k) - \sum_{j=1}^{m} g(p_j)n(\gamma; p_j).
\]
Note. The proof of Theorem V.3.6 is to be given in Exercise V.3.1.

**Proposition V.3.7.** Let \( f \) be analytic on an open set containing \( \overline{B}(a; R) \) and suppose that \( f \) is one to one on \( B(a; R) \). If \( \Omega = f[B(a; R)] \) and \( \gamma \) is the circle \( |z - a| = R \), then \( f^{-1}(\omega) \) is defined for each \( \omega \in \Omega \) by
\[
f^{-1}(\omega) = \frac{1}{2\pi i} \int_{\gamma} \frac{zf'(z)}{f(z) - \omega} \, dz.
\]

Note. We now state as “Rouche’s Theorem” what is actually a generalization of the traditional version (see Conway’s reference at the bottom of page 125).

**Theorem V.3.8. Rouche’s Theorem.**

Suppose \( f \) and \( g \) are meromorphic in a neighborhood of \( \overline{B}(a; R) \) with no zeros or poles on the circle \( \gamma(t) = a + Re^{it}, \, t \in [0, 2\pi] \). Suppose \( Z_f \) and \( Z_g \) are the number of zeros inside \( \gamma \), and \( P_f \) and \( P_g \) are the number of poles inside \( \gamma \) (counted according to their multiplicities) and that \( |f(z) + g(z)| < |f(z)| + |g(z)| \) on \( \gamma \). Then \( Z_f - P_f = Z_g - P_g \).

Note. Rouche’s Theorem can be further generalized by replacing the circle \( \gamma = \{ z \mid |z - a| = R \} \) with any closed rectifiable curve \( \gamma \) where \( \gamma \approx 0 \) in \( G \), and with the introduction of winding numbers (this is Exercise V.3.7).
Note. Ahlfors in his *Complex Analysis* (McGraw Hill, 1979, page 153) state Rouche’s Theorem as:

Let $\gamma \approx 0$ in region $G$ where $n(\gamma; z)$ is either 0 or 1 for any point $z \neq \{\gamma\}$. Let $f$ and $g$ be analytic in $G$ and for all $z \in \{\gamma\}$ suppose $|f(z) - g(z)| < |f(z)|$. Then $f$ and $g$ have the same number of zeros enclosed by $\gamma$.


If $f$ and $g$ are analytic inside and on a simple closed curve $C$ and if $|g(z)| < |f(z)|$ on $C$, then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside $C$.

This version follows from Ahlfors’ version by replacing Ahlfors’ $g(z)$ with $f(z) + g(z)$.

Note. Rouche’s Theorem can be used to give another easy (analytic) proof of the Fundamental Theorem of Algebra.

**Theorem. Fundamental Theorem of Algebra.**

If $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_2z^2 + a_1z + a_0$ is a (complex) polynomial of degree $n$, then $p$ has $n$ zeros (counting multiplicities).

**Proof.** We have

$$
\frac{p(z)}{z^n} = 1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_2}{z^{n-2}} + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}
$$
for $z \neq 0$, and $\lim_{z \to \infty} \frac{p(z)}{z^n} = 1$. So with $\varepsilon = 1$, we have that there exists $R > 0$ such that for all $|z| > R$ we have $\left| \frac{p(z)}{z^n} - 1 \right| < \varepsilon = 1$. That is, for $|z| > R$, $|p(z) - z^n| < |z^n|$. With $f(z) = z^n$ and $g(z) = p(z)$, we have by Ahlfors’ version of Rouche’s Theorem (of course, this also follows from Conway’s version as well) that, since $f(z) = z^n$ has $n$ zeros, then $g(z) = p(z)$ has the same number of zeros.

Revised: 8/8/2017