Chapter VI. The Maximum Modulus Theorem

VI.1. The Maximum Principle

Note. We pick up where we left off in section IV.3 and introduce several more versions of the Maximum Modulus Theorem.

Note. Recall the original version:

Theorem IV.3.11. Maximum Modulus Theorem.
If $G$ is a region and $f : G \to \mathbb{C}$ is an analytic function such that there is a point $a \in G$ with $|f(a)| \geq |f(z)|$ for all $z \in G$, then $f$ is constant.

Let $G$ be a bounded open set in $\mathbb{C}$ and suppose $f$ is a continuous function on $\overline{G}$ which is analytic in $G$. Then

$$\max\{|f(z)| \mid z \in \overline{G}\} = \max\{|f(z)| \mid z \in \partial G\}.$$  

($\overline{G}$ is $G$ closure and $\partial G$ is the boundary of $G$.)
Note. The first version of the Maximum Modulus Theorem applies to an open connected set, and the second version applies to a bounded connected set. Boundedness is necessary in the Maximum Modulus Theorem—Second Version as can be seen by considering set \( G = \{ z \mid \text{Im}(z) \in (-\pi/2, \pi/2) \} \) and \( f(z) = \exp(\exp z) \). Then \( f \) is entire and so analytic and continuous, as needed. If \( z \in \partial(G) \) then \( z = x \pm i\pi/2 \) where \( x \in \mathbb{R} \) and

\[
|f(z)| = |\exp(\exp(x \pm i\pi/2))| = |\exp(\pm ie^x)| = 1.
\]

But 1 is not a bound since \( f \) is unbounded on the real axis \( (f(x) = \exp(\exp x)) \). We can get a third version of the Maximum Modulus Theorem by modifying the boundary concept.

Definition. If \( f : G \to \mathbb{R} \) and \( a \in \overline{G} \) or \( a = \infty \) then the limit superior of \( f(z) \) as \( z \) approaches \( a \) is

\[
\limsup_{z \to a} f(z) = \lim_{r \to 0^+} \left( \sup \{ f(z) \mid z \in G \cap B(a; r) \} \right).
\]

(If \( a = \infty \), \( B(a; r) \) is the ball in the metric of \( \mathbb{C}_\infty \) encountered in section I.6.) Similarly, define

\[
\liminf_{z \to a} f(z) = \lim_{r \to 0^+} \left( \inf \{ f(z) \mid z \in G \cap B(a; r) \} \right).
\]

Definition. If \( G \subseteq \mathbb{C} \) then let \( \partial_\infty(G) \) denote the boundary of \( G \) in \( \mathbb{C}_\infty \), called the extended boundary of \( G \). (Notice that for bounded \( G \), \( \partial_\infty(G) = \partial(G) \), and for unbounded \( G \) \( \partial_\infty(G) = \partial(G) \cup \{ \infty \} \).)

Let $G$ be a region in $\mathbb{C}$ and $f$ an analytic function on $G$. Suppose there is a constant $M$ such that $\limsup_{z \to a} |f(z)| \leq M$ for all $a$ in $\partial_{\infty}(G)$. Then $|f(z)| \leq M$ for all $z$ in $G$.

**Note.** For $f(z) = \exp(\exp z)$ and $G$ as in the above example, $\infty \in \partial_{\infty}(G)$ and $\limsup_{z \to \infty} |f(z)| = \infty$ (let $z$ be real, positive and $z \to \infty$). So the Maximum Modulus Theorem—3rd Version does not apply.

**Note.** We’ll see below that the hypothesis on the behavior of $f$ “at $\infty$” can be weakened (see Theorem VI.1.D).

**Note.** Two simpler versions of the maximum Modulus Theorem—3rd Version (sometimes called the Maximum Modulus Theorem for Unbounded Domains) are the following.

Theorem VI.1.A. Maximum Modulus Theorem for Unbounded Domains (Simplified 1).

Let $R > 0$ and suppose $f$ is analytic on the complement of $\overline{B}(a; R)$, continuous on $\mathbb{C} \setminus B(a; R)$, $|f(z)| \leq M$ on $\partial(B(a; R))$, and $\lim_{|z| \to \infty} |f(z)| \leq M$. Then $|f(z)| \leq M$ on $\mathbb{C} \setminus B(a; R)$. 
Theorem VI.1.B. Maximum Modulus Theorem for Unbounded Domains (Simplified 2).

Let $R > 0$ and suppose $f$ is analytic on the complement of $\overline{B}(a; R)$. Suppose $\lim_{z \to a, |z| > R} |f(z)| \leq M$ for all $|a| = R$ and $\lim_{z \to \infty} |f(z)| \leq M$. Then $|f(z)| \leq M$ on $\mathbb{C} \setminus \overline{B}(a; R)$.

Note. This simplified versions are often useful in applications involving polynomials.

Note. Exercise VI.1.1 requests a proof of the following:

Theorem VI.1.C. Minimum Modulus Theorem. If $f$ is a non-constant analytic function on a bounded open set $G$ and is continuous on $G^-$, then either $f$ has a zero in $G$ or $|f|$ assumes its minimum value on $\partial(G)$.

Note. In the Maximum Modulus Theorem—3rd Version (Theorem VI.1.4), if $\infty \in \partial_{\infty}(G)$ then we required that $\lim_{z \to \infty} |f(z)| \leq M$. In fact, this condition can be weakened. The following is Corollary 1.6.13 from Q. I. Rahman and G. Schemisser’s Analytic Theory of Polynomials, London Mathematical Society Monographs (Book 26), Clarendon Press (2002).

Let $z(t)$, $t \in [\alpha, \beta]$, define a Jordan curves $\Gamma$ with its trace in $\mathbb{C}$ (that is, $\Gamma$ is a simple closed curve in $\mathbb{C}$), and denote the open interior of $\Gamma$ by $\Omega$. Also, let $\varphi$ be a function which is analytic in $\mathbb{C} \setminus \{\Gamma \cup \Omega\}$ and continuous on $\mathbb{C} \setminus \Omega$ such that $|\varphi(z)| \leq M$ for all $z \in \Gamma$. Suppose, in addition, that $\varphi(z)$ tends to a finite limit $\ell$ as $z$ tends to infinity and set $\varphi(\infty) = \ell$. Then, $|\varphi(z)| \leq M$ for all $x$ in $\mathbb{C}_\infty \setminus \{\Gamma \cap \Omega\}$, unless $\varphi$ is a constant.

Note. Of course the hypotheses of the simplified versions of the Maximum Modulus Theorem for Unbounded Domains (Theorems VI.1.B and VI.1.C) can now be weakened by replacing “$\lim_{z \to \infty} |f(z)| \leq M$” with “$\lim_{z \to \infty} |f(z)|$ is finite.”

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