Vector Spaces
Linear Algebra
MATH 2010

• Recall that when we discussed vector addition and scalar multiplication, that there were a set of properties, such as distributive property, associative property, etc. Any set that satisfies these properties is called a vector space and the objects in the set are called vectors.

• Definition of Vector Space: A real vector space is a set of elements \( V \) together with two operations \( \oplus \) and \( \odot \) satisfying the following properties:

A) If \( u \) and \( v \) are any elements of \( V \) then \( u \oplus v \) is in \( V \). (\( V \) is said to be closed under the operation \( \oplus \).)

A1) \( u \oplus v = v \oplus u \) for \( u \) and \( v \) in \( V \). (commutative property)
A2) \( u \oplus (v \oplus w) = (u \oplus v) \oplus w \) for \( u, v, \) and \( w \) in \( V \). (associative property)
A3) There is an element 0, called the zero vector, in \( V \) such that
\[
u \oplus 0 = 0 \oplus u = u
\]
for all \( u \) in \( V \). (additive identity)
A4) For each \( u \) in \( V \), there is an element \( -u \), called the negative of \( u \), in \( V \) such that
\[
u \oplus -u = 0
\]
(additive inverse)

S) If \( u \) is any element of \( V \) and \( c \) is any real number, then \( c \odot u \) is in \( V \). (\( V \) is said to be closed under the operation \( \odot \).)

S1) \( c \odot (u \oplus v) = (c \odot u) \oplus (c \odot v) \) for all real numbers \( c \) and all \( u \) and \( v \) in \( V \). (distributive property)
S2) \( (c + d) \odot u = (c \odot u) \oplus (d \odot u) \) for all real numbers \( c \) and \( d \) and all \( u \) in \( V \). (distributive property)
S3) \( c \odot (d \odot u) = (cd) \odot u \) for all real numbers \( c \) and \( d \) and all \( u \) in \( V \). (associative property)
S4) \( 1 \odot u = u \) for all \( u \) in \( V \). (scalar identity)

• Example: Consider the set \( V \) of all ordered triples of real numbers of the form \((x, y, 0)\) and define the operations \( \oplus \) and \( \odot \) by
\[
(x, y, 0) \oplus (x', y', 0) = (x + x', y + y', 0)
\]
and
\[
c \odot (x, y, 0) = (cx, cy, 0)
\]
Determine whether or not \( V \) is a vector space.

To prove that \( V \) is a vector space, it is necessary to show all 10 properties are satisfied. Intuitive perspective: this is the regular vector addition and scalar multiplication on vectors in which the third component is 0, so one should expect all the properties to be satisfied. Now, you must prove all of them are satisfied.

A) If \( u \) and \( v \) are any elements of \( V \) then \( u \oplus v \) is in \( V \). (\( V \) is said to be closed under the operation \( \oplus \).)

Let \( u = (x, y, 0) \) and \( v = (x', y', 0) \) be two vectors in \( V \). (Note: this is the required form to be in \( V \)). Then
\[
u \oplus v = (x, y, 0) \oplus (x', y', 0) = (x + x', y + y', 0)
\]
by using the definition of \( \oplus \). Since \((x + x', y + y', 0)\) has a third component of 0, it has the right form and is in \( V \). So, property A is satisfied!
A1) \(u \oplus v = v \oplus u\) for \(u\) and \(v\) in \(V\). (commutative property)

Consider the \(u\) and \(v\) from above, i.e, \(u = (x, y, 0)\) and \(v = (x', y', 0)\), as two vectors in \(V\). Then

\[
u \oplus v = (x, y, 0) \oplus (x', y', 0) \quad \text{plugging in } u \text{ and } v
\]

\[
= (x + x', y + y', 0) \quad \text{using the definition of } \oplus
\]

\[
= (x' + x, y' + y, 0) \quad \text{by the commutative property of scalars since } x, x', y, \text{ and } y' \text{ are all scalars}
\]

Similarly,

\[
v \oplus u = (x', y', 0) \oplus (x, y, 0) \quad \text{plugging in } v \text{ and } u
\]

\[
= (x' + x, y' + y, 0) \quad \text{using the definition of } \oplus
\]

Comparing the two above, we have

\[
u \oplus v = v \oplus u
\]

Thus \(A1\) is satisfied.

A2) \(u \oplus (v \oplus w) = (u \oplus v) \oplus w\) for \(u, v,\) and \(w\) in \(V\). (associative property)

Let \(u = (x, y, 0), v = (x', y', 0),\) and \(w = (x'', y'', 0)\) be three vectors in \(V\). Then the left hand side looks like

\[
u \oplus (v \oplus w) = (x, y, 0) \oplus ((x', y', 0) \oplus (x'', y'', 0)) \quad \text{plugging in } u, v, \text{ and } w
\]

\[
= (x, y, 0) \oplus (x' + x'', y' + y'', 0) \quad \text{by definition of } \oplus \text{ on } v \text{ and } w
\]

\[
= (x + (x' + x''), y + (y' + y''), 0) \quad \text{by definition of } \oplus \text{ on } u \text{ and } (x' + x'', y' + y'', 0)
\]

\[
= ((x + x') + x'', (y + y') + y'', 0) \quad \text{by associative property of scalars}
\]

The right hand side is given by

\[
(u \oplus v) \oplus w = ((x, y, 0) \oplus (x', y', 0)) \oplus (x'', y'', 0) \quad \text{plugging in } u, v, \text{ and } w
\]

\[
= (x + x', y + y', 0) \oplus (x'', y'', 0) \quad \text{by definition of } \oplus \text{ on } u \text{ and } v
\]

\[
= ((x + x') + x'', (y + y') + y'', 0) \quad \text{by definition of } \oplus \text{ on } (x + x', y + y', 0) \text{ and } w
\]

Thus, comparing the left side and right side, they are equal. So, \(A2\) is satisfied.

A3) There is an element 0, called the zero vector, in \(V\) such that

\[
u \oplus 0 = 0 \oplus u = u
\]

for all \(u\) in \(V\). (additive identity)

Let \(u = (x, y, 0)\) and \(0 = (0, 0, 0)\). Notice that this is a vector in \(V\) since the last component is a 0 (that is the only requirement to be in \(V\). Then

\[
u \oplus 0 = (x, y, 0) \oplus (0, 0, 0) \quad \text{plugging in } u \text{ and } 0
\]

\[
= (x + 0, y + 0, 0) \quad \text{by definition of } \oplus
\]

\[
= (x, y, 0) \quad \text{by the property of } 0 \text{ in the real numbers}
\]

\[
= u \quad \text{since } u = (x, y, 0)
\]

Thus, \(A3\) is satisfied.
A4) For each \( u \) in \( V \), there is an element \(-u\), called the negative of \( u \), in \( V \) such that
\[
    u \oplus -u = 0
\]
(additive inverse)

Let \(-u = (-x, -y, 0)\) be in \( V \). Then
\[
    u \oplus -u = (x, y, 0) \oplus (-x, -y, 0) \quad \text{by substituting in } u \text{ and } -u
    = (x + (-x), y + (-y), 0) \quad \text{by definition of } \oplus
    = (0, 0, 0) \quad \text{by additive inverse in the real numbers}
\]

Therefore, \textbf{A4 is satisfied}. 

S) If \( u \) is any element of \( V \) and \( c \) is any real number, then \( c \odot u \) is in \( V \). (\( V \) is said to be closed under the operation \( \odot \).)

Let \( u = (x, y, 0) \) be in \( V \). Then
\[
    c \odot u = (cx, cy, 0)
\]
by using the definition of \( \odot \). Since \((cx, cy, 0)\) has a third component of 0, it has the right form and is in \( V \). So, \textbf{property S is satisfied}!

S1) \( c \odot (u \oplus v) = (c \odot u) \oplus (c \odot v) \) for all real numbers \( c \) and all \( u \) and \( v \) in \( V \). (distributive property)

Let \( u = (x, y, 0) \) and \( v = (x', y', 0) \) be in \( V \), then looking at the left hand side above, we have
\[
    c \odot (u \oplus v) = c \odot ((x, y, 0) \oplus (x', y', 0)) \quad \text{substituting in } u \text{ and } v
    = c \odot (x + x', y + y', 0) \quad \text{by definition of } \oplus
    = (c(x + x'), c(y + y'), 0) \quad \text{by definition of } \odot
\]

Now, looking at the right hand side above
\[
    (c \odot u) \oplus (c \odot v) = (c \odot (x, y, 0)) \oplus (c \odot (x', y', 0)) \quad \text{substituting in } u \text{ and } v
    = (cx, cy, 0) \oplus (cx', cy', 0)) \quad \text{by definition of } \odot
    = (cx + cx', cy + cy', 0) \quad \text{by definition of } \oplus
    = (c(x + x'), c(y + y'), 0) \quad \text{by distributive property of real numbers}
\]

Since the left and right hand sides are equal, \textbf{property S1 has been satisfied}.

S2) \( (c + d) \odot u = (c \odot u) \oplus (d \odot u) \) for all real numbers \( c \) and \( d \) and all \( u \) in \( V \). (distributive property)

Let \( u = (x, y, 0) \) be in \( V \). Then the left hand side is
\[
    (c + d) \odot u = (c + d) \odot (x, y, 0) \quad \text{by substituting in } u
    = ((c + d)x, (c + d)y, 0) \quad \text{by definition of } \odot
    = (cx + dx, cy + dy, 0) \quad \text{by distributive property of real numbers}
\]

The right hand side is simply
\[
    (c \odot u) \oplus (d \odot u) = (c \odot (x, y, 0)) \oplus (d \odot (x, y, 0)) \quad \text{by substituting in } u
    = (cx, cy, 0) \oplus (dx, dy, 0) \quad \text{by definition of } \odot
    = (cx + dx, cy + dy, 0) \quad \text{by definition of } \oplus
\]

This is the same as the left hand side, so \textbf{property S2 has been satisfied}.
S3) \( c \odot (d \odot u) = (cd) \odot u \) for all real numbers \( c \) and \( d \) and all \( u \) in \( V \). (associative property)

Again, let \( u = (x, y, 0) \) be in \( V \) and \( c \) and \( d \) be scalars. Then

\[
\begin{align*}
c \odot (d \odot u) &= c \odot (d \odot (x, y, 0)) \quad \text{by substituting in } u \\
&= c \odot (dx, dy, 0) \quad \text{by definition of } \odot \\
&= (c(dx), (cd)y, 0) \quad \text{by definition of } \odot \\
&= ((cd)x, (cd)y, 0) \quad \text{by associative property of real numbers}
\end{align*}
\]

The right hand side

\[
(cd) \odot u = (cd) \odot (x, y, 0) = ((cd)x, (cd)y, 0)
\]

by substitution and definition of \( \odot \). Therefore, property S3 has been satisfied.

S4) \( 1 \odot u = u \) for all \( u \) in \( V \). (scalar identity)

Finally,

\[
1 \odot u = 1 \odot (x, y, 0) = (1x, 1y, 0) = (x, y, 0) = u
\]

by substitution and property of multiplication by 1 for any real number. Therefore, **property S4 has been satisfied** as well.

Since all 10 properties have been proven to be satisfied, \( V \) is said to be a **vector space**.

- **Example**: Consider the set \( V \) of all ordered triples of real numbers of the form \((x, y, z)\) with operations \( \oplus \) and \( \odot \) defined by

\[
(x, y, z) \oplus (x', y', z') = (x + x', y + y', z + z')
\]

and

\[
c \odot (x, y, z) = (cx, y, z)
\]

Determine whether or not \( V \) is a vector space.

To prove that \( V \) is a vector space, it is necessary to show all 10 properties are satisfied. **Intuitive perspective**: this is the regular vector addition but the scalar multiplication is different that standard scalar multiplication. Therefore, one should expect that all the addition properties will be satisfied, but the scalar multiplication properties may or may not be satisfied. If \( V \) is a vector space, then we have to prove that all 10 properties are satisfied. However, if even one property fails, then \( V \) is NOT a vector space. Therefore, it is beneficial to start with the properties that you think may fail. In this case, that would be the scalar multiplication properties.

S) If \( u \) is any element of \( V \) and \( c \) is any real number, then \( c \odot u \) is in \( V \). (\( V \) is said to be closed under the operation \( \odot \).)

Let \( u = (x, y, z) \) be an element of \( V \). (Note: there is no special form to the elements of \( V \) - they are regular vectors of \( \mathbb{R}^3 \).) Then

\[
c \odot (x, y, z) = (cx, y, z)
\]

Since \((cx, y, z)\) is in \( \mathbb{R}^3 \) and there is no special form to vectors in \( V \) other than being in \( \mathbb{R}^3 \), then \( c \odot u \) is in \( V \). Thus **property S is satisfied**.
S1) \( c \odot (u \oplus v) = (c \odot u) \oplus (c \odot v) \) for all real numbers \( c \) and all \( u \) and \( v \) in \( V \). (distributive property)

Let \( u = (x, y, z) \) and \( v = (x', y', z') \) be in \( V \), then

\[
\begin{align*}
    c \odot (u \oplus v) &= c \odot ((x, y, z) \oplus (x', y', z')) \\
                     &= c \odot (x + x', y + y', z + z') & \text{definition of } \oplus \\
                     &= (c(x + x'), y + y', z + z') & \text{definition of } \odot \\
\end{align*}
\]

The right hand side is given by

\[
\begin{align*}
    (c \odot u) \oplus (c \odot v) &= (c \odot (x, y, z)) \oplus (c \odot (x', y', z')) \\
                                &= (cx, y, z) \oplus (cx', y', z') & \text{definition of } \odot \\
                                &= (cx + cx', y + y', z + z') & \text{definition of } \oplus \\
                                &= (c(x + x'), y + y', z + z') & \text{distributive property of real numbers} \\
\end{align*}
\]

The left and right hand side are the same; therefore, property S1 is satisfied.

S2) \( (c + d) \odot u = (c \odot u) \oplus (d \odot u) \) for all real numbers \( c \) and \( d \) and all \( u \) in \( V \). (distributive property)

Let \( u \) be as above and \( c \) and \( d \) be scalars, then the left hand side is given by

\[
\begin{align*}
    (c + d) \odot u &= (c + d) \odot (x, y, z) \\
                   &= ((c + d)x, y, z) & \text{definition of } \odot \\
\end{align*}
\]

The right hand side is given by

\[
\begin{align*}
    (c \odot u) \oplus (d \odot u) &= (c \odot (x, y, z)) \oplus (d \odot (x, y, z)) \\
                               &= (cx, y, z) \oplus (dx, y, z) & \text{definition of } \odot \\
                               &= (cx + dx, y + y, z + z) & \text{definition of } \oplus \\
                               &= ((c + d)x, 2y, 2z) & \text{distributive property of real numbers} \\
\end{align*}
\]

Notice, that the left hand side and right hand side are NOT equal. Thus, property S2 is NOT satisfied.

Since we have found one property which is not satisfied, we do not need to go any further. \( V \) is not a vector space.

- **Important Vector Spaces** There are several important vector spaces worth noting that satisfy all 10 properties.

  1. The set of all \( n \) tuples, \( \mathbb{R}^n \), is a vector space with \( \oplus \) and \( \odot \) defined as the standard operations for addition and scalar multiplication.
  2. The set of all polynomials of degree less than or equal to \( n \), denoted \( P_n \) where \( \oplus \) and \( \odot \) are the standard operations. An element in \( P_n \) has the form

\[
p(t) = a_n t^n + a_{n-1} t^{n-1} + ... + a_1 t + a_0
\]

The zero polynomial is

\[
0 = 0 t^n + 0 t^{n-1} + ... + 0 t + 0
\]

Then if

\[
q(t) = b_n t^n + b_{n-1} t^{n-1} + ... + b_1 t + b_0
\]
the ◦ operation is defined by

\[ p(t) \oplus q(t) = (a_n + b_n)t^n + (a_{n-1} + b_{n-1})t^{n-1} + \ldots + (a_1 + b_1)t + (a_0 + b_0) \]

and the ⊗ operation is given by

\[ c \otimes p(t) = (ca_n)t^n + (ca_{n-1})t^{n-1} + \ldots + (ca_1)t + ca_0 \]

3. The set of all continuous functions on the real number line, denoted \( C(-\infty, \infty) \) (from Calculus) is a vector space with ⊕ given by

\[ (f \oplus g)(x) = f(x) + g(x) \]

and ⊗ is given by

\[ (c \otimes f)(x) = cf(x) \]

where \( f \) and \( g \) are functions in \( C(-\infty, \infty) \) and \( c \) is a scalar. Note that from Calculus, you can recall that the sum of two continuous functions is continuous and the product of a scalar and continuous function is continuous. Also, the zero function is given by

\[ f_0(x) = 0 \text{ for all } x \]

4. The set of all \( m \times n \) matrices, denoted \( M_{m,n} \), is a vector space with typical matrix addition and scalar multiplication.

- **Example:** Let \( V \) be the set of all real numbers with the operations

\[ u \oplus v = u - v \text{ ordinary subtraction} \]

and

\[ c \otimes u = cu \text{ ordinary multiplication} \]

Is \( V \) a vector space? If not, name a property which fails. **Intuition:** if a property will fail, it will probably be ⊕ since this is not the normal definition of addition.

**A)** If \( u \) and \( v \) are any elements of \( V \) then \( u \oplus v \) is in \( V \).

Note that the only requirement to be in \( V \) is to be a real number. Thus, \( u \oplus v = u - v \) is still a real number, so \( V \) is closed under addition and **property A is satisfied.**

**A1)** \( u \oplus v = v \oplus u \) for \( u \) and \( v \) in \( V \). (commutative property)

So, by definition \( u \oplus v = u - v = -v + u = -(v - u) \) by definition of ⊕ and distributive property of real numbers. However, \( v \oplus u = v - u \) by definition of ⊕, so \( u \oplus v \neq v \oplus u \). Property A1 is **NOT satisfied.**

Thus \( V \) is **NOT** a vector space.

- **Example:** Let \( V \) be the set of all 2nd degree polynomials with normal addition and scalar multiplication (not lesser than or equal to 2, but second degree polynomial). Notice, that \( V \) is **NOT** closed under addition. Let

\[ p(t) = a_2t^2 + a_1t + a_0 \]

and

\[ q(t) = b_2t^2 + b_1t + b_0 \]

be in \( V \). Since they are elements of \( V \), \( a_2 \neq 0 \) and \( b_2 \neq 0 \) by the property of \( V \). However,

\[ p(t) \oplus q(t) = (a_2 + b_2)t^2 + (a_1 + b_1)t + (a_0 + b_0) \]

by the standard definition of ⊕ for polynomials. In order for \( V \) to be closed under addition, \( p(t) \oplus q(t) \) must ALWAYS be a second degree polynomial. That means, \( a_2 + b_2 \neq 0 \). However, if \( a_2 = -b_2 \) or \( b_2 = -a_2 \), then \( p(t) \oplus q(t) \) is a first degree polynomial since the coefficient of \( t^2 \) is zero, hence \( V \) is not closed under addition. Therefore, **A is NOT satisfied** and \( V \) is **NOT** a vector space.
• **Theorem:** Let \( v \) be any element of a vector space \( V \) and let \( c \) be any scalar, then the following properties are true:

1. \( 0v = 0 \)
2. \( c0 = 0 \)
3. If \( cv = 0 \), then \( c = 0 \) or \( v = 0 \)
4. \((-1)v = -v\)