Practice Test

Chapter 1

Name ____________________________

Instructions. Show your work and/or explain your answers.

1. Find the length of each vector and the angle between them.

\[ \mathbf{u} = \langle \sqrt{2}, \sqrt{6}, 2\sqrt{2} \rangle \quad \text{and} \quad \mathbf{v} = \langle 0, 0, 1 \rangle \]

Solution: \( \mathbf{u} \cdot \mathbf{u} = 2 + 6 + 8 = 16, \| \mathbf{u} \| = 4, \| \mathbf{v} \| = 1, \mathbf{u} \cdot \mathbf{v} = 2\sqrt{2}, \) so

\[ \cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} \]

Thus, \( \theta = \pi/4. \)

2. Show that if \( \mathbf{u} \) is perpendicular to \( \mathbf{v} \), then

\[ \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2 = \| \mathbf{u} - \mathbf{v} \|^2 \]

Solution: \( \| \mathbf{u} - \mathbf{v} \|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2\mathbf{u} \cdot \mathbf{v} = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2 - 0 \)

3. Find a number \( k \) for which \( \mathbf{u} = \langle 1, 2, 1 \rangle \) is perpendicular to \( \mathbf{v} = \langle k, 3, 4 \rangle \).

Solution: \( \mathbf{u} \cdot \mathbf{v} = k + 6 + 4 = k + 10. \) Thus, \( \mathbf{u} \cdot \mathbf{v} = 0 \) implies that \( k = -10. \)

4. Find the area of the triangle whose vertices are \( P_1 (0, 0, 0), P_2 (1, 1, 0), P_3 (1, 1, 4). \)

Solution: The vectors are \( \mathbf{u} = \langle 1, 1, 0 \rangle \) and \( \mathbf{v} = \langle 1, 1, 4 \rangle \). Their cross product is \( \mathbf{u} \times \mathbf{v} = \langle 4, -4, 0 \rangle, \) so that the area is

\[ A = \frac{1}{2} \| \mathbf{u} \times \mathbf{v} \| = \frac{1}{2} \sqrt{16 + 16 + 0} = 2\sqrt{2} \]

5. Find the equation of the plane through the points \( P_1 (0, 0, 0), P_2 (2, 1, 5), \) and \( P_3 (−1, 1, 2). \)

Solution: The vectors are \( \mathbf{u} = \langle 2, 1, 5 \rangle \) and \( \mathbf{v} = \langle -1, 1, 2 \rangle, \) so that \( \mathbf{u} \times \mathbf{v} = \langle -3, -9, 3 \rangle. \)

Thus, the equation of the plane is

\[ -3(x - 0) - 9(y - 0) + 3(z - 0) = 0 \]

so that in functional form we have \( z = x + 3y. \).
6. Find the \(xy\)-equation and sketch the graph of the curve

\[
\mathbf{r}(t) = \langle \cos(2t), \sin(t) \rangle, \quad t \text{ in } \left[0, \frac{\pi}{2}\right]
\]

**Solution:** Since \(\cos(2t) = 1 - 2\sin^2(t)\), we have \(x = 1 - 2y^2\). Moreover, \(\mathbf{r}(0) = \langle \cos(0), \sin(0) \rangle = (1, 0)\) and \(\mathbf{r}(\pi/2) = \langle \cos(\pi), \sin(\pi/2) \rangle = (-1, 1)\). Thus, the parametrization is the section of the curve \(x = 2y^2 - 1\) from \((1, 0)\) to \((-1, 1)\).

7. Find the cartesian equation of the parametric curve

\[
\mathbf{r}(t) = \langle \sin^2(t), \cos(t) \rangle, \quad t \text{ in } [0, \pi]
\]

Then sketch the curve showing its orientation and its endpoints.

**Solution:** Since \(\cos^2(t) + \sin^2(t) = 1\), we have \(y^2 + x = 1\) or \(x = 1 - y^2\). Moreover, \(\mathbf{r}(0) = \langle 0, 1 \rangle\) and \(\mathbf{r}(\pi) = \langle 0, -1 \rangle\).

8. Find the velocity, speed, and acceleration of the curve with parametrization

\[
\mathbf{r}(t) = \langle t, t^2, t^3 \rangle
\]

**Solution:** The velocity is \(\mathbf{v}(t) = \langle 1, 2t, 3t^2 \rangle\), the acceleration is \(\mathbf{a}(t) = \langle 0, 2, 6t \rangle\), and the speed is

\[
v = \sqrt{1 + 4t^2 + 9t^4}
\]

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9. If a rock is thrown into the air near the earth’s surface with initial velocity \( \mathbf{v}(0) = (16, 0, 64) \) feet per second and initial position \( \mathbf{r}(0) = (0, 0, 6) \) feet, then what is the maximum height of the rock if air resistance is ignored?

**Solution:** Since \( \mathbf{a}(t) = (0, 0, -32) \), we must have \( \mathbf{v}(t) = \int \mathbf{a}(t) \, dt = (0, 0, -32t) + (C_1, C_2, C_3) \). Moreover, \( \mathbf{v}(0) = (16, 0, 64) = (0, 0, 0) + (C_1, C_2, C_3) \), so that \( \mathbf{v}(t) = (16, 0, 64 - 32t) \). Similarly, \( \mathbf{r}(t) = \int \mathbf{v}(t) \, dt \) and the initial condition lead to

\[
\mathbf{r}(t) = \left( 16t, 0, 64t - 16t^2 + 6 \right)
\]

As a result, \( \mathbf{v} \cdot \mathbf{a} = 0 \) when \( 64 - 32t = 0 \), or when \( t = 2 \). Thus, the maximum height is

\[
64 \cdot 2 - 16 \cdot 2^2 + 6 = 70 \text{ feet}
\]

10. The acceleration due to gravity is 12.2 feet per second per second at the surface of Mars. Find the position function \( \mathbf{r}(t) \) of an object with initial velocity \( \mathbf{v}_0 = (30, 0, 40) \) and initial position \( \mathbf{r}_0 = (0, 0, 0) \).

**Solution:** \( \mathbf{v}(t) = \int (0, 0, -12.2) \, dt = (0, 0, -12.2t) + \mathbf{v}_0 = (0, 0, -12.2t) + (30, 0, 40) = (30, 0, 40 - 12.2t) \). Integrating again yields

\[
\mathbf{r}(t) = \int \mathbf{v}(t) \, dt = \int (30, 0, 40 - 12.2t) \, dt = \left( 30t, 0, 40t - 6.1t^2 \right) + \mathbf{r}_0 = \left( 30t, 0, 40t - 6.1t^2 \right)
\]

11. Find the arclength and the unit tangent vector of the curve

\[
\mathbf{r}(t) = (3 \sin(t), 5 \cos(t), 4 \sin(t)), \text{ in } [0, 2\pi]
\]

**Solution:** \( \mathbf{v}(t) = (3 \cos(t), -5 \sin(t), 4 \cos(t)) \), so that the speed is

\[
v = \sqrt{9 \cos^2(t) + 25 \sin^2(t) + 16 \cos^2(t)} = \sqrt{25 \cos^2(t) + 25 \sin^2(t)} = 5
\]

Thus, \( \mathbf{T}(t) = \langle 3/5 \cos(t), -\sin(t), 4/5 \cos(t) \rangle \) and the arclength is

\[
L = \int_0^{2\pi} v \, dt = \int_0^{2\pi} 5 \, dt = 10\pi
\]
12. Find the unit normal $\mathbf{N}$ for the curve

$$\mathbf{r}(t) = \langle \sin(t^3), t^3, \cos(t^3) \rangle$$

**Solution:** To begin with, $\mathbf{v}(t) = \langle 3t^2 \cos(t^3), 3t^2, 3t^2 \sin(t^3) \rangle$, so that the speed is

$$v = \sqrt{9t^4 \cos^2(t^3) + 9t^4 + 9t^4 \sin^2(t^3)} = 3\sqrt{2}t^2$$

Thus, the unit tangent vector is $\mathbf{T}(t) = \langle \cos(t^3) / \sqrt{2}, 1 / \sqrt{2}, \sin(t^3) / \sqrt{2} \rangle$ and

$$\frac{d\mathbf{T}}{dt} = \langle -3t^2 \sin(t^3), 0, -3t^2 \cos(t^3) \rangle$$

from which we find that the normal vector is $\mathbf{N}(t) = \langle -\sin(t^3), 0, -\cos(t^3) \rangle$.

13. Find the arclength of the curve

$$\mathbf{r}(t) = \langle e^{2t}, t, 2e^t \rangle, \quad t \text{ in } [0, 1]$$

**Solution:** $\mathbf{v}(t) = \langle 2e^{2t}, 1, 2e^t \rangle$, so $v = (4e^{4t} + 4e^{2t} + 1)^{1/2} = \left( [2e^{2t} + 1]^2 \right)^{1/2} = 2e^{2t} + 1$.

Thus,

$$L = \int_0^1 v dt = \int_0^1 (2e^{2t} + 1) dt = e^2$$

14. Find the length of the astroid

$$\mathbf{r}(t) = \langle \cos^3(t), \sin^3(t) \rangle, \quad t \text{ in } [0, 2\pi]$$

**Solution:** The velocity is $\mathbf{v} = \langle -3\cos^2(t) \sin(t), 3\sin^2(t) \cos(t) \rangle$, so that the speed is

$$v = \sqrt{9 \cos^4(t) \sin^2(t) + 9 \sin^4(t) \cos^2(t)} = \sqrt{9 \cos^2(t) \sin^2(t) \left( \cos^2(t) + \sin^2(t) \right)}$$

Thus, the speed is $v = 3 |\cos(t) \sin(t)|$, so that the arclength is

$$L = \int_0^{2\pi} v dt = 4 \int_0^{\pi/2} 3 \sin(t) \cos(t) dt = 6$$

15. Find the curvature of the curve

$$\mathbf{r}(t) = \langle \sin(t), \cos(t), \ln|\sec(t)| \rangle$$

**Solution:** The velocity is $\mathbf{v}(t) = \langle \cos(t), -\sin(t), \tan(t) \rangle$, which implies that the speed is

$$v = \sqrt{\cos^2(t) + \sin^2(t) + \tan^2(t)} = \sqrt{1 + \tan^2(t)} = \sqrt{\sec^2(t)}$$
Thus, $v = \sec(t)$, which implies that the unit tangent vector is

$$T(t) = \cos(t) \langle \cos(t), -\sin(t), \tan(t) \rangle = \langle \cos^2(t), -\sin(t) \cos(t), \sin(t) \rangle$$

It then follows that the derivative of the unit tangent is

$$\frac{dT}{dt} = \langle 2 \cos(t) \sin(t), \sin^2(t) - \cos^2(t), \cos(t) \rangle = \langle \sin(2t), -\cos(2t), \cos(t) \rangle$$

As a result, the curvature is

$$\kappa = \frac{1}{\sec(t)} \sqrt{\sin^2(2t) + \cos^2(2t) + \cos^2(t)} = \cos(t) \sqrt{1 + \cos^2(t)}$$