Practice Test

Chapter 2

Name ________________________________

Instructions. Show your work and/or explain your answers.

1. Find the domain of the function

   \[ f(x, y) = \sqrt{y} + \sqrt{x^2 - 1} \]

   Is the domain open, closed, or neither? Bounded or unbounded? Connected or not connected?

   **Solution:** \( \text{dom}(f) = \{(x, y) \mid y \geq 0 \text{ and } x^2 \geq 1\} = \{(x, y) \mid x \leq -1, \ y \geq 0\} \cup \{(x, y) \mid x \geq 1, \ y \geq 0\} \)

   Since domain contains its boundaries \( y = 0, \ x = -1, \) and \( x = 1, \) it is closed. Since \( x \) and \( y \) can approach infinity within the domain, it is unbounded. Since no path exists from \( \{(x, y) \mid x \leq -1, \ y \geq 0\} \) to \( \{(x, y) \mid x \geq 1, \ y \geq 0\} \) that stays in the domain, the domain is not connected.

2. Show the following limit does not exist by showing that different paths through the origin lead to different limits:

   \[ \lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2-y^2} \]

   **Solution:** Along \( y = 0, \) the limit is 1. Along \( x = 0, \) the limit is \(-1\).

3. Does the following limit exist?

   \[ \lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2} \]

   **Solution:** No. Along \( x = 0 \) and \( y = 0, \) the limit is 1. However, along \( y = x \) the limit is 2.

4. Find the linearization of \( f(x, y) = x + e^{xy} \) at \((1,0)\)

   **Solution:** \( f_x(x, y) = 1 + ye^{xy}, \ f_y(x, y) = xe^{xy} \). Thus, \( L(x, y) = 2 + 1(x - 1) + 1(y - 1) \).

5. Find the second order derivatives of

   \[ f(x, y) = x^2 + e^{xy} \]

   **Solution:** \( f_x = 2x + ye^{xy}, \ f_y = xe^{xy}, \ f_{xx} = 2 + y^2 e^{xy}, \ f_{yy} = x^2 e^{xy}, \ f_{xy} = e^{xy} + xe^{xy} \).
6. Find the separated solution of
\[
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = u
\]

**Solution:** \(u(x, t) = \phi(x) T(t)\) implies that \(\phi T' + \phi' T = \phi T\), so that
\[
\frac{T'}{T} + \frac{\phi'}{\phi} = 1, \quad \frac{T'}{T(t)} = \frac{\phi'(x)}{\phi(x)} + 1
\]

Thus, \(T'(t) = -kT(t)\) and \(\phi'(x) = (-1 - k) \phi(x)\), so that the separated solution is
\[
\phi(x) T(t) = Pe^{-kt}e^{(-k-1)x}
\]

7. Find \(\partial_u z\) when \(z = x^2 + y^3\) and \(x = u^2 + uv, y = u^3v\)

**Solution:** The chain rule implies that
\[
\frac{\partial z}{\partial u} = 2x \frac{\partial x}{\partial u} + 3y \frac{\partial y}{\partial u}
\]
\[
= 2 \left( u^2 + uv \right) \left( 2u + v \right) + 3 \left( u^3v \right) \left( 3u^2v \right)
\]
\[
= 4u^3 + 6u^2v + 2uv^2 + 9u^8v^3
\]

8. Prove that the derivative of a sum is the sum of the derivatives by applying the chain rule for 2 variables to
\[
w = x + y
\]

where \(x = f(t)\) and \(y = g(t)\).

**Solution:** The chain rule implies that
\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}
\]

However, \(w_x = 1\) and \(w_y = 1\), and also \(w = f(t) + g(t)\), so that
\[
\frac{d}{dt} (f(t) + g(t)) = \frac{dw}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = f'(t) + g'(t)
\]

thus completing the proof.

9. Find the gradient of the function \(g(x, y) = x^2 + y^2\), and then show that it is normal to the curve \(x^2 + y^2 = 25\) at the point \((3, 4)\).

**Solution:** The curve \(x^2 + y^2 = 25\) is a circle. \(\nabla g = \langle 2x, 2y \rangle\), so \(\nabla g(3, 4) = \langle 6, 8 \rangle\). However, \(\nabla g(3, 4) = \langle 6, 8 \rangle\) is parallel to the radius \((3, 4)\) and thus must be perpendicular to the tangent line.
10. In what direction is the function $f(x, y) = x^2 + y^3$ decreasing the fastest at the point (1, 3)?

**Solution:** The gradient of $f$ is $\nabla f = \langle 2x, 3y^2 \rangle$, so that $\nabla f (1, 3) = \langle 2, 27 \rangle$. This is the direction in which $f$ is increasing the fastest. The direction $f$ is decreasing the fastest is thus

$$-\nabla f (1, 3) = \langle -2, -27 \rangle$$

11. Find the extrema and saddle points of $f(x, y) = x^2 + 3xy + 2y^2 - 4x - 5y$.

**Solution:** $f_x = 2x + 3y - 4$, $f_y = 3x + 4y - 5$. Thus,

$$2x + 3y = 4$$

$$3x + 4y = 5$$

Thus, the critical point is $(-1, 2)$. Moreover, $f_{xx} = 2$, $f_{xy} = 3$, and $f_{yy} = 4$, so that

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 4 \cdot 2 - 3^2 = -1$$

and thus there is a critical point at the saddle.

12. Find the extrema and saddle points of $f(x, y) = 4x^3 - 6x^2y + 3y^2$

**Solution:** $f_x = 12x^2 - 12xy$, $f_y = -6x^2 + 6y$, Thus, $12x^2 = 12xy$ and $6x^2 = 6y$. Since $x^2 = xy$, $x = 0$ or $x = y$. If $x = 0$, then $x^2 = y$ implies that $y = 0$, and the critical point is $(0, 0)$. If $x = y$, then $x^2 = y$ implies that $x^2 = x$ or $x = 1, 0$. Thus, the critical points are $(0, 0)$ and $(1, 1)$. However,

$$D = (24x - 12y)6 - (12x)^2 = 144x - 72y - 144x^2$$

Thus, $D(0, 0) = 0$ and there is no info, and $D(1, 1) = 144 - 72 - 144 = -72 < 0$, so there is a saddle at $(1, 1)$.

13. Find the point(s) on the curve $xy = 1$ that are closest to the origin.

**Solution:** That is, minimize $f(x, y) = x^2 + y^2$ subject to $xy = 1$. If $g(x, y) = xy$, then $\nabla f = \langle 2x, 2y \rangle$ and $\nabla g = \langle y, x \rangle$, so that

$$2x = \lambda y, \quad 2y = \lambda x$$

Since neither $x, y$ can be zero, we have $\lambda = 2x/y$, so that

$$2y = \frac{2x}{y} x \quad y^2 = x^2 \quad y = x, y = -x$$

If $y = -x$, then $xy = -x^2 = 1$ which has no solution. If $y = x$, then $xy = x^2 = 1$, so $x = 1, -1$ and the critical points are $(1, 1)$ and $(-1, -1)$. In both cases $f(1, 1) = f(-1, -1) = 2.$ Moreover, $f(2, 1/2) = 4.25$, so we must have minima at these points.
14. Use Lagrange Multipliers to solve the following: John wants to build a 500 $ft^2$ deck behind his house.

His house is 50 feet long, and correspondingly, he wants the deck to be between 5 and 50 feet long. What dimensions of the deck will minimize the lengths of the rail around the 3 exposed sides of the deck?

**Solution:** Let $x$ be the length and $y$ be the width of the deck. Then $xy = 500$. Let $L$ denote the length of the rail. Then

$$L = x + 2y$$

Thus, we must minimize $L = x + 2y$ subject to $xy = 500$ for $x$ in $[5, 50]$. If $g(x, y) = xy$, then $\nabla L = \langle 1, 2 \rangle$ and $\nabla g = \langle y, x \rangle$, so that

$$1 = \lambda y, \quad 2 = \lambda x$$

Since $\lambda = 1/y$, substitution leads to

$$2 = \frac{1}{y} x \quad \text{and} \quad 2y = x$$

Substituting $x = 2y$ into the constraint yields $2y^2 = 500$, or

$$y^2 = 250, \quad y = \sqrt{250} = 5\sqrt{10}$$

If $x = 2y$ and $y = 5\sqrt{10}$, then $x = 10\sqrt{10}$, so that $\left(10\sqrt{10}, 5\sqrt{10}\right)$ is a critical point. At that point

$$L = 10\sqrt{10} + 2 \cdot 5\sqrt{10} = 20\sqrt{10} = 63.25'$$

When $x = 5$, then $y = 100$ and at $(5, 100)$ the length is

$$L = 5 + 2 \cdot 100 = 205$$

When $x = 50$, then $y = 10$ and at $(50, 10)$, the length is

$$L = 50 + 2 \cdot 10 = 70'$$

Thus, the shortest rail occurs when $x = 10\sqrt{10} = 31.622'$ and $y = 5\sqrt{10} = 15.811'$
15. ** Heating of a 2 dimensional surface (such as in a sheet of metal) is modeled by the 2 dimensional heat equation

\[
\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2} + k^2 \frac{\partial^2 u}{\partial y^2}
\]

where \( u(x, y, t) \) is a function of 3 variables and \( k \) is a constant. What is the separable solution of the 2 dimensional heat equation (hint: involves 2 separation constants)?

**Solution:** Let \( u(x, y, t) = \phi(x) \rho(y) T(t) \). Then

\[
\phi(x) \rho(y) T'(t) = k^2 \phi''(x) \rho(y) T(t) + k^2 \phi(x) \rho''(y) T(t)
\]

Dividing through by \( \phi(x) \rho(y) T(t) \) leads to

\[
\frac{\phi(x) \rho(y) T'(t)}{\phi(x) \rho(y) T(t)} = \frac{k^2 \phi''(x) \rho(y) T(t)}{\phi(x) \rho(y) T(t)} + \frac{k^2 \phi(x) \rho''(y) T(t)}{\phi(x) \rho(y) T(t)}
\]

which simplifies to

\[
\frac{T'(t)}{T(t)} = \frac{k^2 \phi''(x)}{\phi(x)} + \frac{k^2 \rho''(y)}{\rho(y)}
\]

Both sides of the equation must be constant, so that

\[
\frac{T'(t)}{T(t)} = -\omega^2 \quad \text{and} \quad \frac{k^2 \phi''(x)}{\phi(x)} + \frac{k^2 \rho''(y)}{\rho(y)} = -\omega^2
\]

The second equation can now be written as

\[
\frac{k^2 \phi''(x)}{\phi(x)} = -\omega^2 - \frac{k^2 \rho''(y)}{\rho(y)}
\]

thus implying both sides of this equation are constant (we let \(-\lambda^2\) denote this constant).

\[
\frac{k^2 \phi''(x)}{\phi(x)} = -\lambda^2 \quad \text{and} \quad -\omega^2 - \frac{k^2 \rho''(y)}{\rho(y)} = -\lambda^2
\]

The last equation becomes

\[
\omega^2 \rho(y) + k^2 \rho''(y) = \lambda^2 \rho(y) \quad \text{or} \quad \rho''(y) + \frac{\omega^2 - \lambda^2}{k^2} \rho(y) = 0
\]

If \( \omega^2 > \lambda^2 \), then the equation is a harmonic oscillator and has a solution of

\[
\rho(y) = A_1 \cos \left( \frac{y \sqrt{\omega^2 - \lambda^2}}{k} \right) + B_1 \sin \left( \frac{y \sqrt{\omega^2 - \lambda^2}}{k} \right)
\]

If \( \omega^2 = \lambda^2 \), then \( \rho''(y) = 0 \) and \( \rho(y) = A_1 + B_1 y \). If \( \omega^2 < \lambda^2 \), then

\[
\rho(y) = A_1 \cosh \left( \frac{y \sqrt{\omega^2 - \lambda^2}}{k} \right) + B_1 \sinh \left( \frac{y \sqrt{\omega^2 - \lambda^2}}{k} \right)
\]
Moreover,

\[ \frac{k^2 \phi''(x)}{\phi(x)} = -\lambda^2 \] implies that \( \phi''(x) + \frac{\lambda^2}{k^2} \phi(x) = 0 \)

implies that

\[ \phi(x) = \rho(y) = A_2 \cos \left( \frac{\lambda}{k} x \right) + B_2 \sin \left( \frac{\lambda}{k} x \right) \]

and finally, \( T'(t) = -\omega^2 T(t) \) implies that \( T(t) = P e^{-\omega^2 t} \). Thus, for \( \omega^2 > \lambda^2 \), the separated solution is

\[ u(x, y, t) = P e^{-\omega^2 t} \left( A_2 \cos \left( \frac{\lambda}{k} x \right) + B_2 \sin \left( \frac{\lambda}{k} x \right) \right) \left( A_1 \cos \left( \frac{y \sqrt{\omega^2 - \lambda^2}}{k} \right) + B_1 \sin \left( \frac{y \sqrt{\omega^2 - \lambda^2}}{k} \right) \right) \]

and similar for the other two cases.