The Fifteen Puzzle
A Motivating Example for the Alternating Group
(Supplemental Material for Intro to Modern Algebra)

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1 Introduction

In the infamous "Fifteen Puzzle," numbered tiles are slid either horizontally or vertically into an empty slot. Folklore tells us that in 1886, puzzle master Sam Loyd offered a one-thousand dollar prize if anyone could swap tiles 14 and 15 and return the other tiles to their original slots. However, Slocum's book [6] argues that the puzzle is much older. Regardless of the origins of the puzzle, we wish to know if this is possible. Fortunately, the tools of modern algebra will help us to unravel this puzzle. Our treatment of the puzzle will follow many of the articles written on this puzzle such as [1, 4, 5, 7].

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

$\overset{?}{\rightarrow}$

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

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2 Results about Permutations from Modern Algebra

Recall that a *permutation* on a (finite) set S is a bijection that maps S to itself. Typically, we think of a permutation as a function that rearranges or shuffles the elements of S. Thus, any possible (or impossible) state of the Fifteen Puzzle is simply a permutation of the tiles. Without loss of generality, we can assume that any permutation on the tiles returns the "blank" tile to the lower right corner. As usual, we can represent any state of the Fifteen Puzzle using our "two-line notation" for permutations. In this case, we can use the top line or the array to represent the "slot" and the second line of the array to represent the tile currently occupying the slot. For example,

11	14	10	6
9	4	12	5
15	8	3	13
2	1	7	

can be represented as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 11 & 14 & 10 & 6 & 9 & 4 & 12 & 5 & 15 & 8 & 3 & 13 & 2 & 1 & 7 \end{pmatrix}.$$

Remark 2.1 *There are two main conventions form multiplying permutation. Some authors such as Herstein [3] multiply permutations from left to right. Others such as Fraleigh [2] multiply permutations from* right to left. *For consistency, we will adopt the convention used in Fraleigh.*

Recall that every permutation can represented as a product of disjoint cycles. Note that these disjoint cycles can be written in any order as disjoint cycles commute. So the above permutation can represented more compactly using the *orbits* (or cycles) of the elements. By convention, algebraists typically omit any point that is mapped to itself (i.e., we omit the *fixed points* of the permutation). In our example, slot 1 now has tile 11. Slot 11 now has tile 3. Likewise, slot 3 now has tile 10, and so on. This gives us the cycle notation for the above permutation:

$$(1, 11, 3, 10, 8, 5, 9, 15, 7, 12, 13, 2, 14)(4, 6).$$

Recall that a *transposition* is a 2-cycle. Further, we can write any cycle as product of transpositions. For example,

$$(1, 2, 3, 4, 5) = (1, 5)(1, 4)(1, 3)(1, 2).$$

In particular, the permutation in our example can be written as

$$(1, 11, 3, 10, 8, 5, 9, 15, 7, 12, 13, 2, 14)(4, 6) =$$

$$(1, 14)(1, 2)(1, 13)(1, 12)(1, 7)(1, 15)(1, 9)(1, 5)(1, 8)(1, 10)(1, 3)(1, 11)(4, 6).$$

In this case, our permutation has been written as a product of thirteen transpositions. Likewise, the permutation that flips tiles 14 and 15 while returning all other tiles to their original positions can be written as $(14, 15)$, which is a single transposition.

Any permutation can be written as either

(i) A product of an even number of transpositions *or*

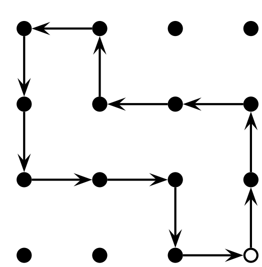
(ii) A product of an odd number of transpositions.

Note that *no* permutation can be written as *both* a product of an even number of transpositions *and* a product of an odd number of transpositions.

Recall that the set of *all* permutations on n symbols is the nth *Symmetric group*, denoted S_n. Clearly, the set of permutations on the tiles of the Fifteen Puzzle is a subgroup of S_{15}. Whereas the subgroup of S_n consisting of *only* the even permutations is known as the *Alternating group* on n symbols. The nth Alternating group is denoted A_n.

Note that the permutation $(14, 15)$ is an odd permutation. Hence, if we can show that the group of permutations on the tiles of the Fifteen Puzzle is a subgroup of the Alternating group, then swapping those tiles as Loyd described is impossible.

Consider the set of permutations in which the blank tile is returned to its original position. In particular, look at paths that start and end with the blank tile in the bottom right corner. One such circuitous path is as follows:



Observe that in this circuit:

(i) For every up, there is a down.

(ii) For every left, there is a right.

Thus, we must have an even number of transpositions. Thus the next proposition follows immediately.

Proposition 2.2 *The set of permutations on the tiles of the Fifteen Puzzle that return the blank tile to the bottom right corner is a subgroup of the Alternating group, A_{15}.*

Hence, swapping tiles 14 and 15 while returning all other tiles to their original position is impossible!

A natural question is whether every even permutation is obtainable as a permutation of the tiles of the Fifteen Puzzle. In other words, is the set of permutations on the tiles of the Fifteen Puzzle in which the blank tile is returned to the bottom right corner isomorphic to A_{15}? In order to answer this question, we will need a few facts about the Alternating group.

3 True facts about the Alternating group

Recall that if S is a *generating set* of a group G, then every element of G can be written as a product of the elements of S. Typically, we think of the elements of S as an "alphabet." If S is a generating set, then every element in G can be written as a "word," where the letters of the word come from our alphabet. So, how do we generate the Alternating group?

Proposition 3.1 *The Alternating group A_n, $n \geq 3$, is generated by the set of 3-cycles. In other words,*

$$A_n = \langle (a, b, c) : a, b, c \in [n], a \neq b, a \neq c, b \neq c \rangle.$$

Proof. Note that $A_3 \cong \mathbb{Z}_3$, so the claim clearly holds. We may then assume that $n \geq 4$.

Take a, b, c, d to be distinct elements of $[n]$. Note that a pair of transpositions will share at most two of these elements. We need only show how to represent every possible pair of transpositions (up to relabeling the letters) can be represented as a product of three cycles. The possibilities are:

$$e = (a,b)(a,b) = (a,b,c)^3,$$

$$(a,c)(a,b) = (a,b,c),$$
$$(a,b)(a,c) = (a,c,b),$$
$$(a,b)(c,d) = (a,c,b)(a,c,d).$$

■

Showing that we can get *every* possible 3-cycle as a permutation could be both difficult and time consuming. For this reason, our next proposition provides a more useful way of generating the Alternating group.

Proposition 3.2 *Let a and b be distinct, fixed (but arbitrary) elements of $[n]$, where $n \geq 3$. The Alternating group A_n is generated by 3-cycles of the form (a,b,k), where $k \in [n] - \{a,b\}$. In other words,*

$$A_n = \langle (a,b,k) : k \in [n] - \{a,b\} \rangle.$$

Proof. By Proposition 3.1, A_n is generated by the set of all possible 3-cycles. Thus, it is sufficient to show that we can represent any 3-cycle as a product of 3-cycles of the form (a,b,k). Note that any 3-cycle will contain at most two of the elements of $\{a,b\}$. For this reason, the possibilities are:

$$(a,k,b) = (a,b,k)^{-1} = (a,b,k)^2,$$

$$(a,c,d) = (a,b,d)(a,b,c)^2,$$
$$(b,c,d) = (a,b,d)^2(a,b,c),$$
$$\begin{aligned}(c,d,e) &= (a,b,c)^2(a,b,e)(a,b,d)^2(a,b,c)\\ &= (a,b,c)^{-1}\left((a,b,e)(a,b,d)^{-1}\right)(a,b,c).\end{aligned}$$

■

In our next section, we will use these results to prove that set of permutations on the tiles of the Fifteen Puzzle is isomorphic to the Alternating group A_{15}.

4 The Puzzle Group for the Fifteen Puzzle

In order to show our main result, we only need to show that we can get the 3-cycles described in Proposition 3.2 as a permutation of the tiles of the Fifteen Puzzle. First we examine a permutation of the tiles in the bottom right corner.

Lemma 4.1 *The set of permutations of the Fifteen Puzzle contains the 3-cycle* $\tau = (11, 12, 15)$.

Proof. In its natural state, tiles in the bottom right corner are:

11	12
15	

Shift tile 15 to the right. This gives the following configuration:

11	12
	15

Shift tile 11 down. This gives the following configuration:

	12
11	15

Shift tile 12 left. This gives the following configuration:

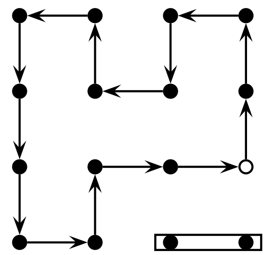
12	
11	15

Shift tile 15 up. This gives the following configuration:

12	15
11	

Hence, we have obtained the permutation $(11, 12, 15)$. ■

For the remaining 3-cycles, we will construct a "long cycle" that leaves two elements (say 11 and 12) and passes through the remaining elements. Such a cycle would look like:



Our next lemma explicitly describes how to achieve this "long cycle."

Lemma 4.2 *The cycle* $\rho = (1, 2, 6, 7, 3, 4, 8, 15, 10, 14, 13, 9, 5)$ *is a permutation of the tiles of the Fifteen Puzzle.*

Proof. Begin by moving 12 down, 11 right, 15 up, 12 left, and 11 down. The bottom right corner now has the following configuration:

15	
11	12

Note that at this point, all other tiles are still in their original positions. We now perform the "long cycle" described above as follows: move 15 right, 10 right, 14 up, 13 right, 9 down, 5 down, 1 down, 2 left, 6 up, 7 left, 3 down, 4 left, 8 up, and 15 up. Currently, the tiles are in the following configuration:

2	6	4	8
1	7	3	15
5	14	10	
9	13	12	11

Finally, we move 11 up, 12 right, 10 down, 11 left, and 12 up. This results in the following configuration:

2	6	4	8
1	7	3	15
5	14	11	12
9	13	10	

This is the permutation $(1, 2, 6, 7, 3, 4, 8, 15, 10, 14, 13, 9, 5)$. ∎

Notice that if we apply ρ once, then tile 10 is in slot 15. If we apply ρ enough times, we can move any of the tiles in the set $[15] - \{11, 12\}$ into the slot 15. For instance, if we wished to have tile 3 in slot 15, then we would apply ρ ten times. This action is typically represented by ρ^{10}.

Our goal is to show that every permutation in A_{15} is a permutation on the tiles of the Fifteen Puzzle. To do this, we let 11 and 12 fulfill the roles of a and b, respectively, in Proposition 3.2. Thus, for $k \in [15] - \{11, 12\}$, we need only show that the permutation $(11, 12, k)$ is a permutation of the tiles of the Fifteen Puzzle.

To do this, we take advantage of the concept of *conjugacy.* Recall that elements a and b are *conjugate elements* in the group G if there is an element $g \in G$ such that $g^{-1}ag = b$. Our strategy will be to show that any 3-cycle of the form $(11, 12, k)$ is conjugate with the 3-cycle $(11, 12, 15)$ (recall that we showed $(11, 12, 15)$ is possible in Lemma 4.1). To do this, we will use ρ^m (where ρ is defined in Lemma 4.2) to position our target tile k in slot 15. We then apply the permutation $(11, 12, 15)$ to rotate tiles 11, 12, and the target. Finally, we apply the permutation ρ^{-m} to return all tiles except 11, 12, and k to their original positions.

For example, to obtain the permutation $(11, 12, 8) = (8, 11, 12)$, we will apply ρ, then τ, and finally ρ^{-1}. This can be represented symbolically as

$$\begin{gathered}\rho^{-1}\tau\rho = \\ (1,2,6,7,3,4,8,15,10,14,13,9,5)^{-1}(11,12,15)(1,2,6,7,3,4,8,15,10,14,13,9,5) \\ = (1,5,9,13,14,10,15,8,4,3,7,6,2)(11,12,15)(1,2,6,7,3,4,8,15,10,14,13,9,5) \\ = (8,11,12).\end{gathered}$$

Theorem 4.3 *The set of permutations on the tiles of the Fifteen Puzzle is isomorphic to A_{15}, the Alternating group on 15 symbols.*

Proof. By Lemma 4.1 and Lemma 4.2, the permutations $\tau = (11, 12, 15)$ and $\rho = (1, 2, 6, 7, 3, 4, 8, 15, 10, 14, 13, 9, 5)$ are permutations of the tiles of the Fifteen Puzzle. Suppose that m is the smallest positive integer such that ρ^m maps tile k to slot 15. We first apply ρ^m in order to position tile k in slot 15. We then apply τ to rotate tiles 11, 12, and 15. We then apply ρ^{-m} to return all tiles (except 11, 12, and k to their original positions). This results in the permutation $(11, 12, k)$. Since k is arbitrary, we can generate A_{15} by Proposition 3.2. ■

In our next section, we consider a generalization of the Fifteen Puzzle in which there are n rows and m columns.

5 Generalizations of the Fifteen Puzzle

Suppose that we have a sliding tile puzzle in a n by m rectangular grid. Again, there is one empty slot and the tiles can move horizontally or vertically. Let $P_{n,m}$ denote the group of permutations on the tiles of this puzzle. Using the same argument as in Proposition 2.2, the following proposition is immediate.

Proposition 5.1 *The group $P_{n,m}$ is a subgroup of the Alternating group A_{nm-1}.*

Note that in its original configuration, the tile in the ith row and jth column is $(i-1)n+j$. Note that we can get a specific 3-cycle $(nm-m-1, nm-m, nm-1)$ using the same argument as in Lemma 4.1.

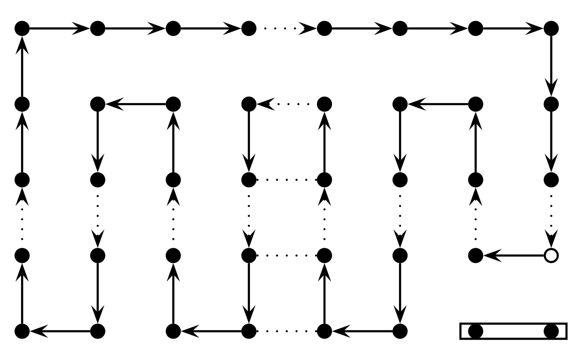
Lemma 5.2 *The group $P(n,m)$ contains the 3-cycle $(nm-m-1, nm-m, nm-1)$.*

Proof. Move $nm-1$ right, $nm-m-1$ down, $nm-m$ left, and $nm-1$ up. ∎

Lemma 5.3 *If m is even, then $P_{n,m}$ contains the cycle $\sigma = (nm-1, a_1, ..., a_{nm-4})$, where $\{a_1, ..., a_{nm-4}\} = [nm-1] - \{nm-m-1, nm-m, nm-1\}$.*

Proof.

Note that we can rotate the elements $nm-m-1, nm-m, nm-1$ so that $nm-1$ is in slot $nm-m-1$ and the blank tile is in slot $nm-m$ as in Lemma 4.2. It suffices to give a "long cycle" that passes through each tile (other than $nm-m-1$ and $nm-m$) exactly once. Once such cycle is given below:



We then rotate the three tiles in the bottom right corner so that tiles $nm-m-1$ and $nm-m$ have been returned to their original position. ■

Lemma 5.4 *If m is even, then $P_{n,m}$ contains the 3-cycle $(nm-m-1, nm-m, a_k)$, where $a_k \in [nm-1] - \{nm-m-1, nm-m\}$.*

Proof. By Lemma 5.2 and Lemma 5.3 the permutations $(nm-m-1, nm-m, nm-1)$ and $\sigma = (nm-1, a_1, ..., a_{nm-4})$, where $\{a_1, ..., a_{nm-4}\} = [nm-1] - \{nm-m-1, nm-m, nm-1\}$. Note that σ^k maps tile a_k to slot $nm-1$. Thus,

$$(nm-m-1, nm-m, a_k) = \sigma^{-k}(nm-m-1, nm-m, nm-1)\sigma^k.$$

■

Lemma 5.4 in fact proves that when m is even, $P_{n,m} \cong A_{nm-1}$. To prove the case when n and m are both odd, we would hope to find a "long cycle" such as in Lemma 5.3. Unfortunately, such a "long cycle"is impossible when n and m are both odd, as will be discussed in the next section. However, we can again use the idea of conjugacy to get the remaining cases.

Theorem 5.5 *The puzzle group $P_{n,m}$ is isomorphic to the isomorphic to the Alternating group, A_{nm-1}.*

Proof. If n or m is even, then the result follows from Proposition 3.2 and Lemma 5.4. Thus, we may assume that n and m are both odd.

Consider the tiles in columns 2,...,m. Since the number of columns in this subgrid is even, every permutation of the form $(nm-m-1, nm-m, k)$, where $k \neq 1 \pmod{m}$ is possible. In particular, the permutation $(nm-m-1, nm-m, 2)$ is possible.

Slide the tiles $nm-1$, $nm-2$,...,$nm-m+2$ to the right. Notice that the blank tile is now in slot $nm-m+2$. Moreover, the first two columns now form a $P_{n,2}$. Thus, we can get every 3-cycle on these tiles by the above arguments. In particular, we can get every permutation of the form $(2, 2+m, 1+km)$, where $0 \leq k \leq n-1$. Now,

$$(nm-m-1, nm-m, 1+km) =$$

$$(2, 2+m, 1+km)^{-1}(nm-m-1, nm-m, 2)(2, 2+m, 1+km)$$
$$= (2, 1+km, 2+m)(nm-m-1, nm-m, 2)(2, 2+m, 1+km).$$

Again, this allows us to generate the Alternating group A_{nm-1} by Proposition 3.2. ■

6 Hamilton Cycles and Bipartite Graphs

In the previous section, we made note of the fact that it was impossible to create a "long cycle" on the n by m grid, when n and m are both odd. In this section, we prove this. To do this, we will need a few elementary concepts from graph theory. For a more comprehensive introduction to graph theory, refer to West [8].

If you have taken graph theory course, then you know that a graph is a collection of *vertices* (usually, these are visually represented as points in the plane) and *edges* between pairs of distinct vertices. If G is a graph, then the vertex set of G is denoted $V(G)$. Likewise, the edge set of G is denoted $E(G)$. If an edge uv is between two vertices u and v, then we say that u and v are *adjacent*.

We now describe the rectangular play field of the Fifteen Puzzle in terms of graphs. Begin by assigning a vertex for each slot in the Fifteen Puzzle. The slot in the ith row and jth column will be associated with vertex (i, j). So,

$$V(G) = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Two vertices $(i, j), (r, s) \in V(G)$ are adjacent if and only if either

(i) $i - r = \pm 1$ and $j = s$ *or*

(ii) $i = r$ and $j - s = \pm 1$.

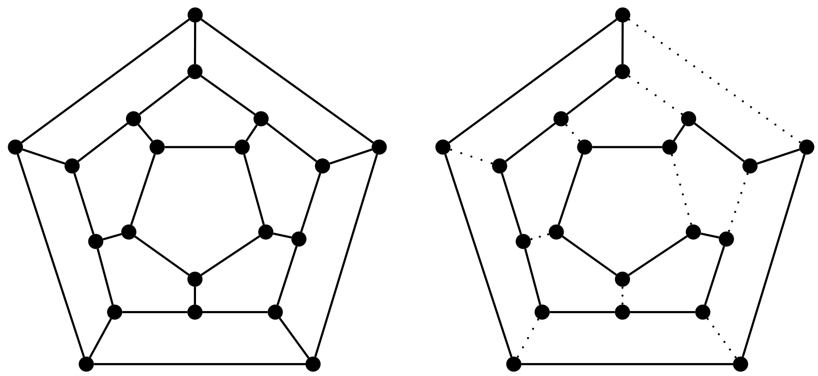
In graph theory, such a graph on n rows and m columns is called a *mesh* and is denoted $M_{n,m}$.

We say that G is a *bipartite graph* if $V(G) = X \cup Y$ such that $X \cap Y = \emptyset$ and if $xy \in E(G)$, then $x \in X$ and $y \in Y$. In our next result, we show that all meshes are bipartite graphs. To do this, we basically place the slots in a "checker board" pattern.

Proposition 6.1 *The mesh $M_{n,m}$ is a bipartite graph.*

Proof. We need only assign vertices to the sets X and Y. We assign vertex (i, j) to set X if $i + j$ is even. Likewise, if $i + j$ is odd, we assign (i, j) to set Y. Suppose that $(i, j), (r, s) \in X$. Since $i + j$ and $r + s$ are both even then either $|i - r| \geq 2$ or $|j - s| \geq 2$. In either case, the two vertices cannot be adjacent. A similar argument holds if both vertices are in Y. ■

In graph theory, these "long cycles" are often described in terms of *hamilton cycles*. A hamilton cycle on a graph is a cycle that passes through every vertex once and exactly once and returns to the original vertex. As usual, the edges of the hamilton cycle must be edges in the original graph. These cycles are named in honor of William Rowan Hamilton who developed the *icosian game*. An example of a hamilton cycle is given on the icosian game below.



In the last section, we gave a partial result on the existence of a hamilton cycle on the mesh $M_{n,m}$, where at least one of n or m is even.

Proposition 6.2 *Suppose m is even. The mesh $M_{n,m}$ has a hamilton cycle.*

Proof. Since m is even, we write it as $m = 2t$. We use a "saw tooth" pattern as in the previous section. The explicit pattern is to start at $(2, 1)$:

(i) For $i = 1, ..., t$, traverse $(j, 2i - 1)$, where $j = 2, ..., n$, then traverse $(n - p + 2, 2i)$, where $p = 2, ..., n$.

(ii) Next, traverse $(1, 2t - q + 1)$, where $q = 1, ..., 2t$.

(iii) Finally, return to $(2, 1)$.

∎

We finally, show our main result for this section. Namely, we show that if n and m are both odd, then $M_{n,m}$ has no hamilton cycle. Actually, we show a more general result regarding bipartite graphs.

Proposition 6.3 *Let G be a bipartite graph with $V(G) = X \cup Y$. If $|X| \neq |Y|$, then G has no hamilton cycle.*

Proof. Any hamilton cycle on a bipartite graph must alternate between vertices in X and vertices in Y. Since $|X| \neq |Y|$, this is impossible. ∎

In particular, using the "checker board" pattern described in Proposition 6.1 on $M_{2n+1,2m+1}$ results in $|X| = 2nm+n+m+1$ and $|Y| = 2nm+n+m$. Hence, there is no hamilton cycle on $M_{2n+1,2m+1}$.

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