Instant Insanity (Supplemental Material for Intro to Graph Theory)

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1 Introduction

Instant Insanity (see Figure 1) is a puzzle introduced around 1900 when it was called 'The Great Tantalizer' (or simply the Tantalizer). It gained popularity in the 1960's because of a version manufactured by Parker Brothers. It is a puzzle consisting of four cubes. Each of the six faces of each cube is colored with one of four colors: Blue, Green, Red, or White. The goal is to stack the four cubes on top of each other such that each color appears exactly once on each of the four sides of the resulting tower. Our treatment of the Instant Insanity puzzle will follow the numerous mathematical sources such as [3, 7, 9, 12, 15]. A version of Instant Insanity on other Platonic solids was studied in [11]. There is a "sequel" puzzle, Instant Insanity II (see Figure 2), that was studied in [2, 13].

There are several versions of the puzzle. Each appears to be identical to the version I purchased, up to permutations of the colors. The cubes in the version I purchased can be described using the *net* of the cube. The net of a solid is obtained by "unfolding" the sides of the solid so that each face shares a border with at least one of its previous neighbors. The result can easily be represented in the plane. For example, one possible net of the cube is given in Figure 3.

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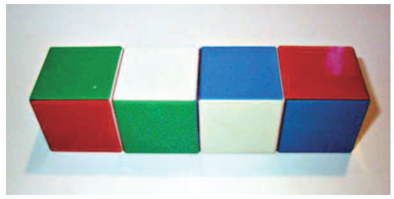

Figure 1: Instant Insanity

Figure 2: Instant Insanity II

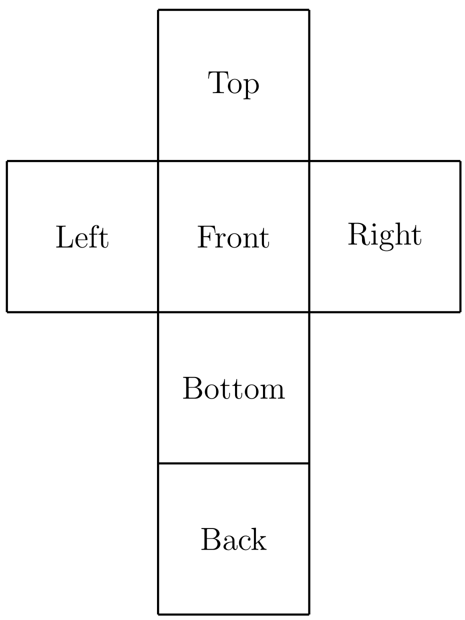

Figure 3: A net of the cube

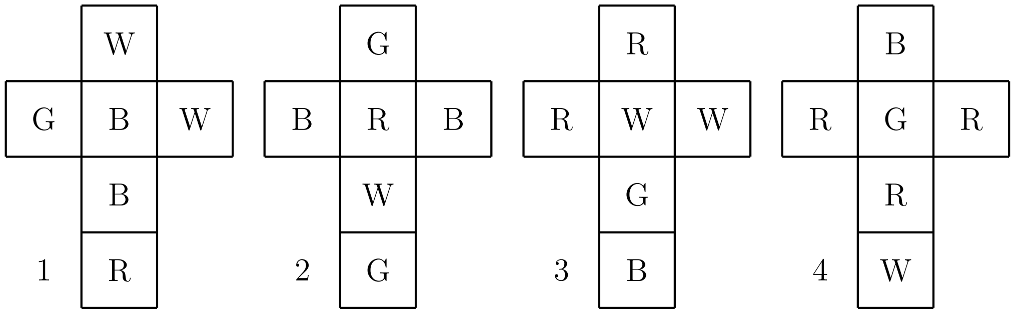

Figure 4: The cubes for Instant Insanity

Using the net in Figure 3 and the first letter for each color, we can represent each of the cubes in the puzzle. This is given in Figure 4.

2 Number of States

One way to measure the difficulty of a puzzle is to determine the possible number of states. In this case, we want to know the number of possible ways to arrange the cubes. This number can be determined using elementary combinatorics. For a more comprehensive introduction to combinatorics, refer to [1, 8, 14]. First, we need to know how many ways each cube can be rotated. This is simply the number of elements in the rotation group of the cube. See [5, 6, 10] for more information on group theory.

Proposition 2.1 *There are 24 ways to rotate the cube. Equivalently, there are 24 elements in the rotational group of the cube.*

With Proposition 2.1 in mind, we are now prepared to compute the number of states of Instant Insanity.

Theorem 2.2 *There are 41472 states of the Instant Insanity puzzle, up to rotating and flipping the tower or permuting the order of the cubes.*

Proof. We begin by determining the number of states when rotations, flips, and permutations of the cubes are considered distinct. This can be done by:

(i) Ordering the four cubes. There are 4! ways to do this.

(ii) Rotating the four cubes individually. Each cube has 24 possible rotations by Proposition 2.1. Thus there are 24^4 ways to rotate the cubes.

It follows from the Multiplication Principle that there are $4! * 24^4 = 7962624$ states when rotations, flips, and permutations of the cubes are considered distinct.

To obtain the number of states when rotations, flips, and permutations of the cubes are not considered distinct, we simply divide by this number. There are:

(i) There are 4 ways to rotate the tower.

(ii) There are 2 ways to flip the tower.

(iii) There are 4! ways to permute the cubes.

Thus, up to rotations, flips, and permutations of the cubes, the number of states is given by

$$\frac{4! * 24^4}{4 * 2 * 4!} = 41472.$$

■

3 Solution

To determine a solution to Instant Insanity, we will construct a graph for each of the four cubes. The vertices of each graph will be the four colors. We will connect two (not necessarily distinct) vertices when their corresponding color is on opposite faces of the cube (e.g., Front and Back are opposite faces of the cube). Further, these edges will be oriented so that Left points to Right, Front points to Back, and Top points to Bottom. These graphs are given in Figure 5. For more information on graph theory, see [4, 16].

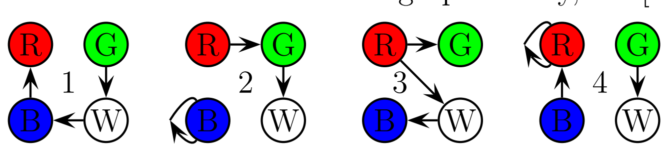

Figure 5: Graphs for the four cubes

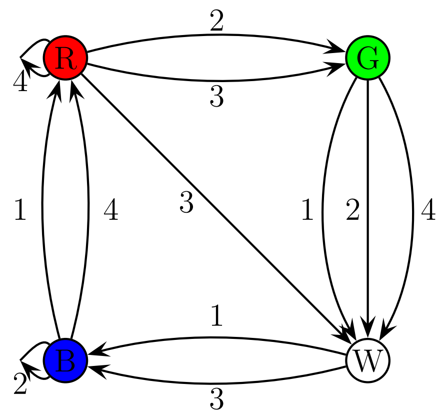

Figure 6: The combined graph

We now combine these graphs into a single multigraph. The edges are labeled with the number of the cube they came from. The result is given in Figure 6.

Our goal is to find two directed cycles within the multigraph from Figure 6. The first of these cycles will determine the Left and Right faces of the completed tower. The second cycle will determine the Front and Back faces of the completed tower. Hence, these cycles must satisfy:

(i) Each cycle passes through each vertex exactly once. In other words, these are hamilton cycles.

(ii) Each cycle uses an edge from each cube exactly once.

(iii) No edge is on both cycles.

Note that the edge labeled 1 from B to R must be on one of the cycles, say the Left/Right cycle. Since each cycle uses an edge from a cube exactly once, the edge labeled 3 from W to B must also be on this cycle. Likewise, since 3 cannot be repeated, the edge labeled 2 from R to G must be on the Left/Right cycle. By process of elimination, the edge labeled 4 from G to W must be on this cycle as well. The arcs for the Front/Back cycle are obtained in a similar manner. This results in the cycles shown in Figure 7.

From these cycles, we can obtain the solution of Instant Insanity as follows:

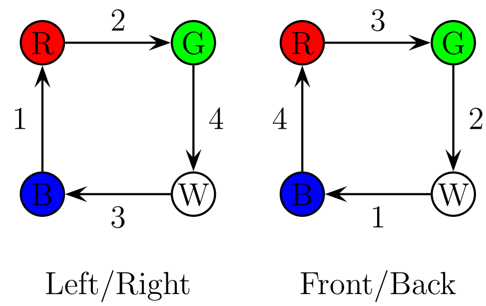

Figure 7: The two cycles

(i) Cube 1: Blue points to Red on the Left/Right cycle and White points to Blue on the Front/Back cycle. Thus, we orient Cube 1 so that its Left face is Blue, its Right face is Red, its Front face is White, and its Back face is Blue.

(ii) Cube 2: Red points to Green on the Left/Right cycle and Green points to White on the Front/Back cycle. Thus, we orient Cube 2 so that its Left face is Red, its Right face is Green, its Front face is Green, and its Back face is White.

(iii) Cube 3: White points to Blue on the Left/Right cycle and Red points to Green on the Front/Back cycle. Thus, we orient Cube 3 so that its Left face is White, its Right face is Blue, its Front face is Red, and its Back face is Green.

(iv) Cube 4: Green points to White on the Left/Right cycle and Blue points to Red on the Front/Back cycle. Thus, we orient Cube 4 so that its Left face is Green, its Right face is White, its Front face is Blue, and its Back face is Red.

This solution is summarized in Table 1.

A natural question is to whether the solution given in Table 1 is unique. Theorem 3.1 shows that we have a unique solution.

Theorem 3.1 *There is a unique solution to the Instant Insanity puzzle, up to rotations, flips, and permutations of the cubes.*

Proof. A solution to Instant Insanity is given in Table 1.

Cube	Left	Front	Right	Back
1	Blue	White	Red	Blue
2	Red	Green	Green	White
3	White	Red	Blue	Green
4	Green	Blue	White	Red

Table 1: The solution for Instant Insanity

To show uniqueness, consider the multigraph in Figure 6. We must construct two cycles as in Figure 7. For purposes of exposition, we represent each edge as an ordered triple (a, b, c), where a is the number of the cube and b and c are the endpoints of the edge. Note that to construct our cycles, we can take none of the edges $(2, B, B)$, $(3, R, W)$, and $(4, R, R)$. Hence $(1, B, R)$ must be on one cycle while $(4, B, R)$ will be on the other. Suppose that $(1, B, R)$ is on the first cycle. Since two edges cannot come from Cube 1, this cycle must then contain $(3, B, W)$. Likewise, we cannot take two edges from Cube 3, so we must use the edge $(2, G, R)$. Using a similar argument, we must include the edge $(4, G, W)$ on the first cycle. Using a similar argument, the second cycle must include the edges $(1, B, W)$, $(2, G, W)$, $(3, G, R)$, and $(4, B, R)$. Ergo, the solution is unique. ∎

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