From WINNING WAYS FOR YOUR MATHEMATICAL PLAYS, VOL 4 BY BERLEKAMP, CONWAY, & GUY

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Pursuing Puzzles Purposefully

The chapter of accidents is the longest chapter in the book.
John Wilkes

I shall proceed to such Recreations as adorn the Mind;
of which those of the Mathematicks are inferior to none.
William Leybourne; *Pleasure with Profit.*

We know you want to use your winning ways mostly when playing with other people, but there are quite a lot of puzzles that are so interesting that you really feel you're playing a game against some invisible opponent—perhaps the puzzle's designer—maybe a malevolent deity. In this chapter we'll discuss a few cases where some kind of strategic thinking simplifies the problem. But because we don't want to spoil your fun we'll try to arrange not always to give the *whole* game away.

Soma

This elegant little puzzle was devised by Piet Hein. Figure 1 shows the seven non-convex shapes that can be made by sticking 4 or fewer $1 \times 1 \times 1$ cubes together. Piet Hein's puzzle is to assemble these as a $3 \times 3 \times 3$ cube.

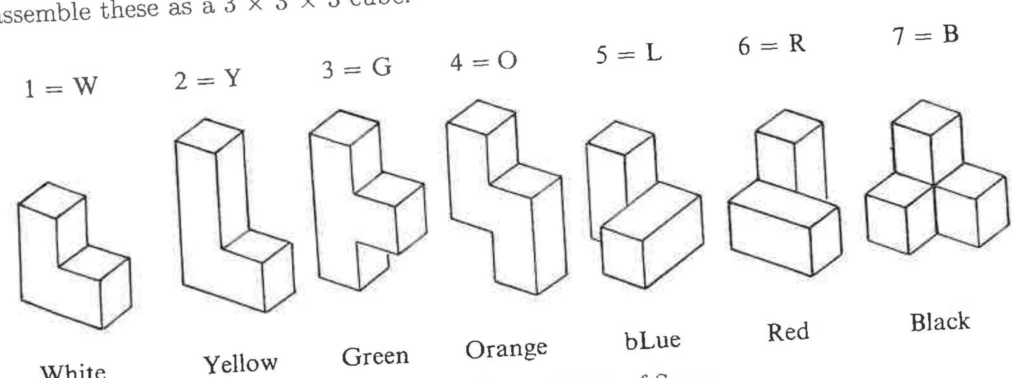


Figure 1. The Seven Pieces of Soma.

We advise you to use seven different colors for your pieces as in the figure. Many people solve this puzzle in under ten minutes, so it can't be terribly hard. But we've got a distinct feeling that it's much harder than it ought to be. Is this just because the pieces have such awkwardly wriggly shapes?

Blocks-in-a-Box

Here is another puzzle invented by one of us some years ago, in which all the pieces are rectangular cuboids but it still seems undeservedly hard to fit them together. We are asked to pack one 2 × 2 × 2 *cube*, one 2 × 2 × 1 *square*, three 3 × 1 × 1 *rods* and thirteen 4 × 2 × 1 *planks* into a 5 × 5 × 5 box (Fig. 2). It's quite easy to get all but one of the blocks into the box, but somehow one piece always seems to stick out somewhere. A friend of ours once spent many evenings without ever finding a solution. Why is it so much harder than it seems to be?

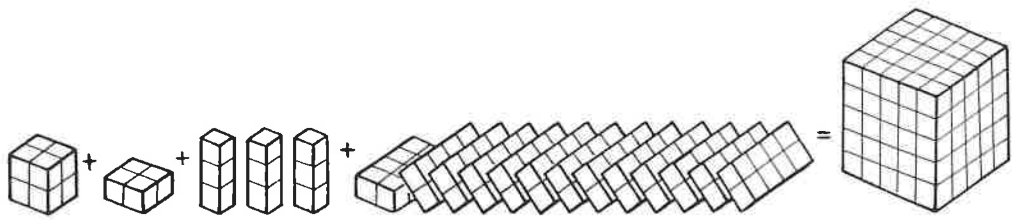


Figure 2. The Eighteen Pieces for Blocks-in-a-Box.

Hidden Secrets

In our view the good puzzles are those with simple pieces but difficult solutions. Anyone can make a hard puzzle with lots of complicated pieces but how can you possibly make a hard puzzle out of a few easy pieces?

When a seemingly simple puzzle is unexpectedly difficult, it's usually because, as well as the obvious problem, there are some hidden ones to be attended to. Both Soma and Blocks-in-a-Box have such hidden secrets, but let's look at a much simpler puzzle, to fit six 2 × 2 × 1 squares into a 3 × 3 × 3 box, leaving three of the 1 × 1 × 1 cells empty—the *holes* (Fig. 3). This now seems fairly trivial, but even so there's a hidden secret which sometimes makes people take more than 5 minutes over it. This hidden problem comes from the fact

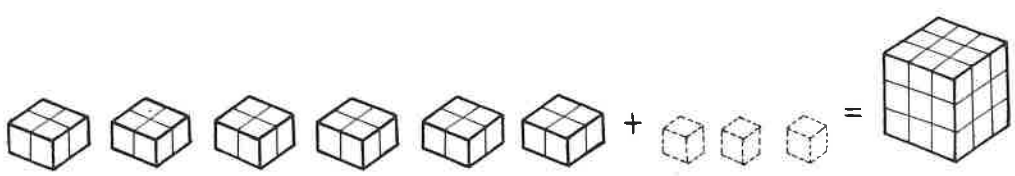


Figure 3. A Much Simpler Puzzle.

that the square pieces can only occupy an even number of the cells in each horizontal layer. So since 9 is odd each horizontal layer must have a hole and there are only just enough holes to go round. Of course these holes must also manage to meet each of the three layers in each of two vertical directions—you can't afford to have two holes in any layer, because some other layers would have to go without.

So the problem wasn't really to fit the *pieces* in but rather the *holes*. Only when you've realized this do you see why the unique solution (Fig. 4) has to be so awkward looking, with the holes strung out in a line between opposite corners rather than neatly arranged at the top of the box.

Perhaps you'd like to try the big Blocks-in-a-Box problem now, before looking at the extra hints in the Extras.

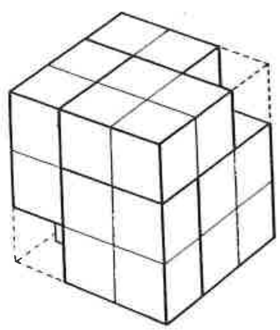


Figure 4. Six Squares in a 3 × 3 × 3 Box.

The Hidden Secrets of Soma

It's because the Soma puzzle pieces have to satisfy some hidden constraints as well as the obvious ones, that it causes most people more trouble than it should. Let's see why.

The 3 × 3 × 3 cube has 8 *vertex* cells, 12 *edge* cells, 6 *face* cells and 1 *central* cell as in Fig. 5.

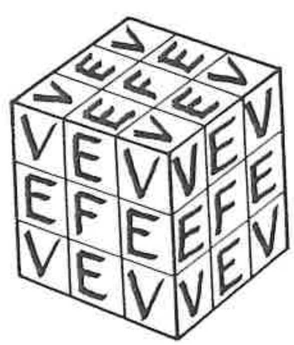


Figure 5. The Vertex, Edge, Face and Central (invisible) Cells.

Now the respective pieces can occupy at most

W	Y	G	O	L	R	B
1,	2,	2,	1,	1,	1,	1

of the vertex cells, so just one piece, the **deficient** one, must occupy just one less vertex-cell than it might. The green piece can't be deficient without being doubly so, and therefore:

> the Green piece has its spine along an edge of the cube.

Now let's color the 27 cells of the cube in two alternating colors,

Flame for the 14 FaVored cells, F and V,
Emerald for the 13 ExCeeded ones, E and C.

Then in *one* solution that we know, the respective pieces occupy

W	Y	G	O	L	R	B	
2	+2	+3	+2	+2	+2	+1	= 14 F, V cells,
1	+2	+1	+2	+2	+2	+3	= 13 E, C cells,

but the Yellow, Orange, bLue and Red pieces, and we now know also the Green piece, *must* occupy these numbers in *every* solution, and therefore so must the White and the Black, since an interchange of colors in either or both of these would alter the totals.

> The White piece occupies 2 FV cells, 1 EC cell.

> The Black piece occupies 1 FV cell and 3 EC ones.

For the placing of a single piece within the box, these considerations leave only the positions of Fig. 6 (which all arise). You'll see that up to symmetries of the cube, the placement of any single piece is determined by whether or not it is deficient and whether or not it occupies the central cell.

The hidden secrets of Soma make it quite likely that one of the first few pieces you put in may already be wrong, when of course you'll spend a lot of time assembling more pieces before such a mistake shows its effect. This would happen for instance if you started by putting the

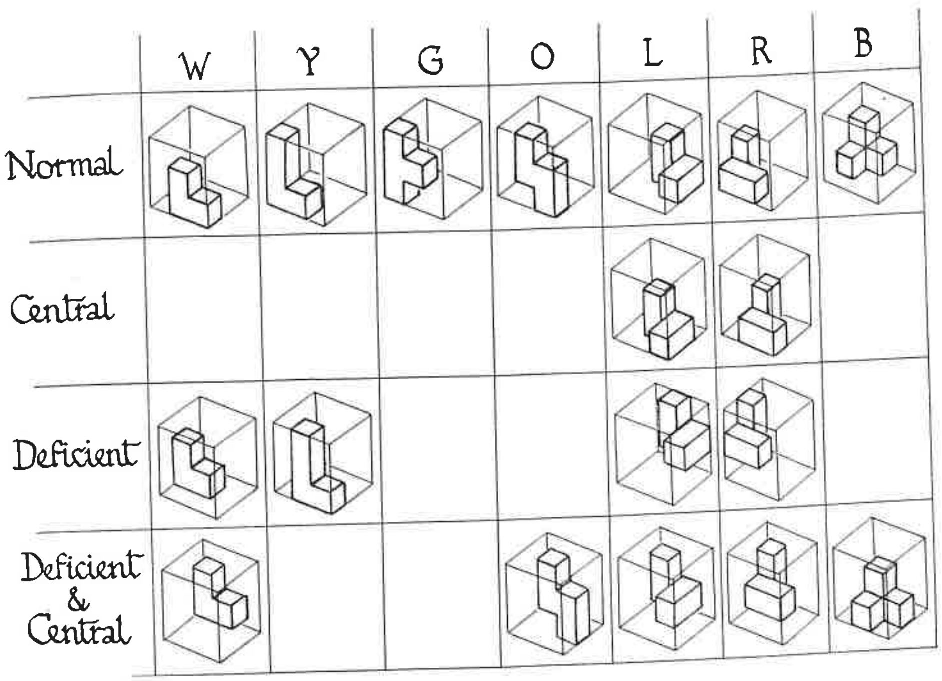


Figure 6. All Possible Positions for the Seven Soma Pieces.

corner of the White piece into a corner of the cube. But if you only put the pieces into the allowed positions, you'll find a solution almost as soon as you start. The complete list of 240 Soma solutions was made by hand by J.H. Conway and M.J.T. Guy one particularly rainy afternoon in 1961. The SOMAP in the Extras enables you to get to 239 of them, when you've found one—*and* located it on the map!

Hoffman's Arithmetico-Geometric Puzzle

A well-known mathematical theorem is the inequality between the arithmetic and geometric means:

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Figure 7 provides a neat proof of this in the form

$$4ab \leq (a+b)^2$$

and the three variable version

$$27abc \leq (a+b+c)^3$$

has prompted Dean Hoffman to enquire whether 27 $a \times b \times c$ blocks can always be fitted into a cube of side $a+b+c$. This turns out to be quite a hard puzzle if a, b, c are fairly close

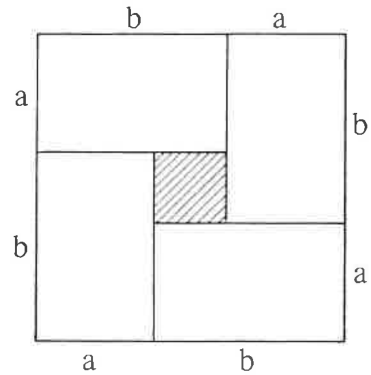


Figure 7. Proof of the Arithmetico-Geometric Inequality.

together but not equal. A good practical problem is to fit

$$27\ \ 4 \times 5 \times 6 \text{ blocks into a } 15 \times 15 \times 15 \text{ box.}$$

With these choices, as for any others with

$$\frac{1}{4}(a+b+c) < a < b < c,$$

it can be shown that each vertical stack of three blocks must contain just one of each height a, b, c, while there must be just three of each height in each horizontal layer. There must be the same unused area on each face (just 3 square units in the $4 \times 5 \times 6$ case).

It's almost impossible to solve the puzzle if you don't keep these hidden secrets constantly in mind because you'll make irretrievable mistakes like making a stack of three height 5 blocks, or leaving a 2×2 empty hole on some face. When you *do* keep them in mind, the puzzle becomes much easier, being only extremely difficult! You'll find some information about solutions to Hoffman's puzzle in the Extras.

Coloring Three-by-Threee-by-Three by Three, Bar Three

In Hoffman's $3 \times 3 \times 3$ puzzle, the three lengths along any line of three had to be different. Can you color the cells of a $3 \times 3 \times 3$ tic-tac-toe board with

three different colors,

using all

three colors the same

number (9) of times, in such a way that *none* of the $\frac{1}{2}(5^3 - 3^3) = 49$ tic-tac-toe lines uses

three different colors,

nor has all its

three colors the same?

Extras

Blocks-in-a-Box

The key to this puzzle is that every piece except the three $3 \times 1 \times 1$ rods occupies as many "black" cells as "white" in every layer. The rods must therefore be arranged so as to correct the color compositions in all fifteen layers simultaneously. It turns out that there is a unique arrangement which does this. Figure 61 also shows the only three dispositions for the $2 \times 2 \times 2$ cube and $2 \times 2 \times 1$ square. With these five pieces in place, the puzzle becomes easy.

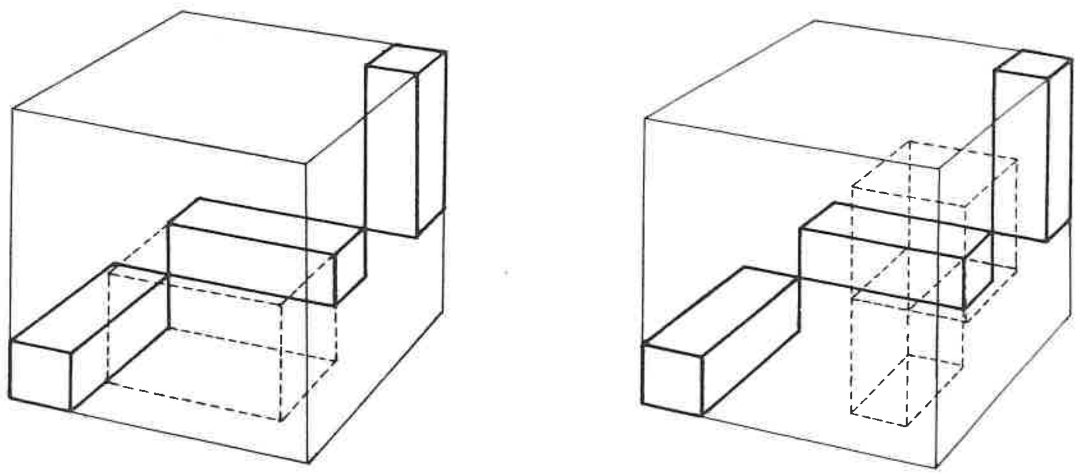
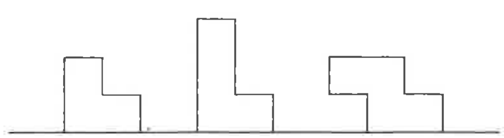


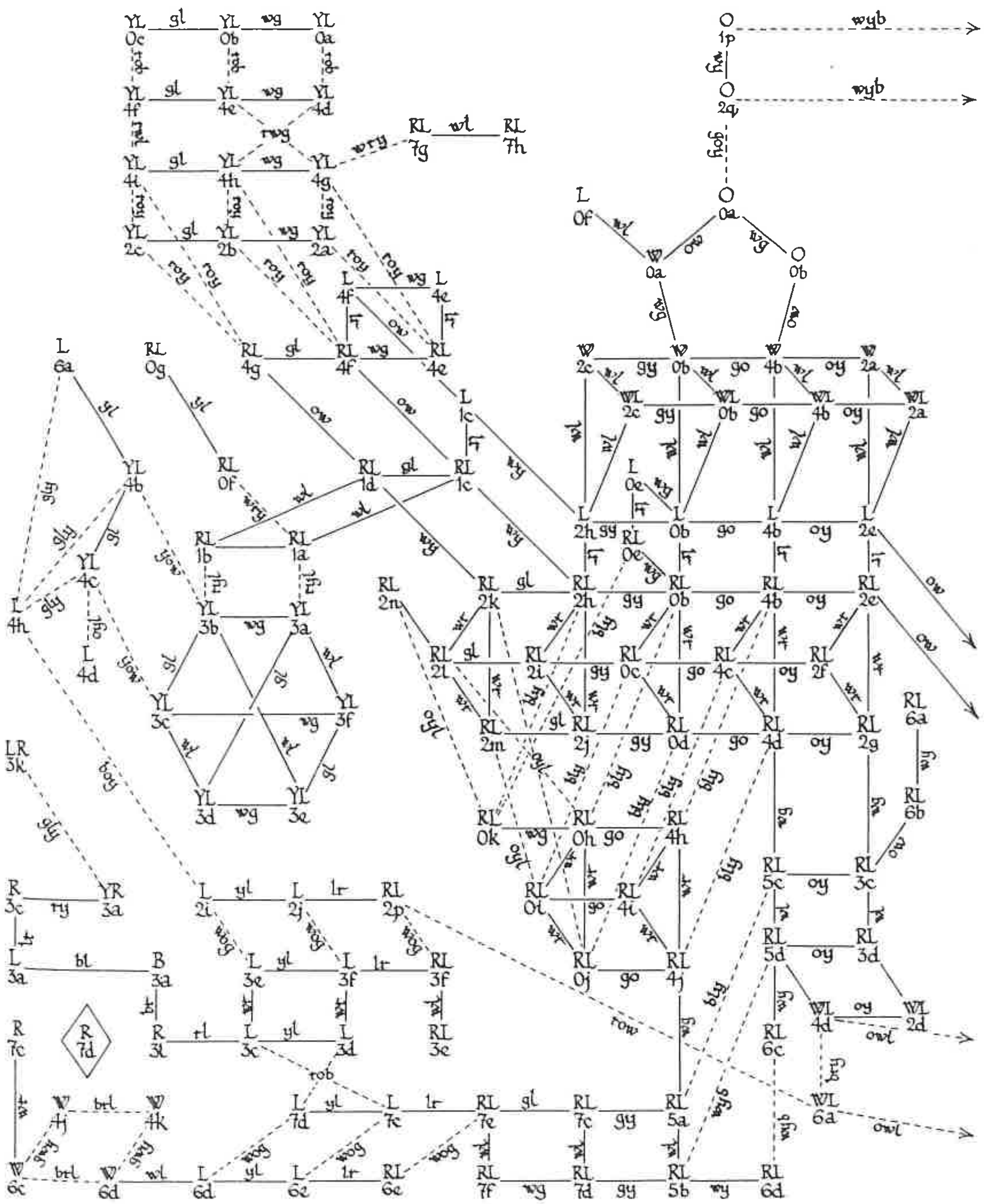
Figure 61. Were You Able to Fit the Blocks-in-a-Box?

A much harder puzzle is to pack 41 $1 \times 2 \times 4$ planks (together with 15 $1 \times 1 \times 1$ holes) into a $7 \times 7 \times 7$ box (see reference to Foregger, and to Mather, who proves that 42 planks can't be packed.)

The Somap

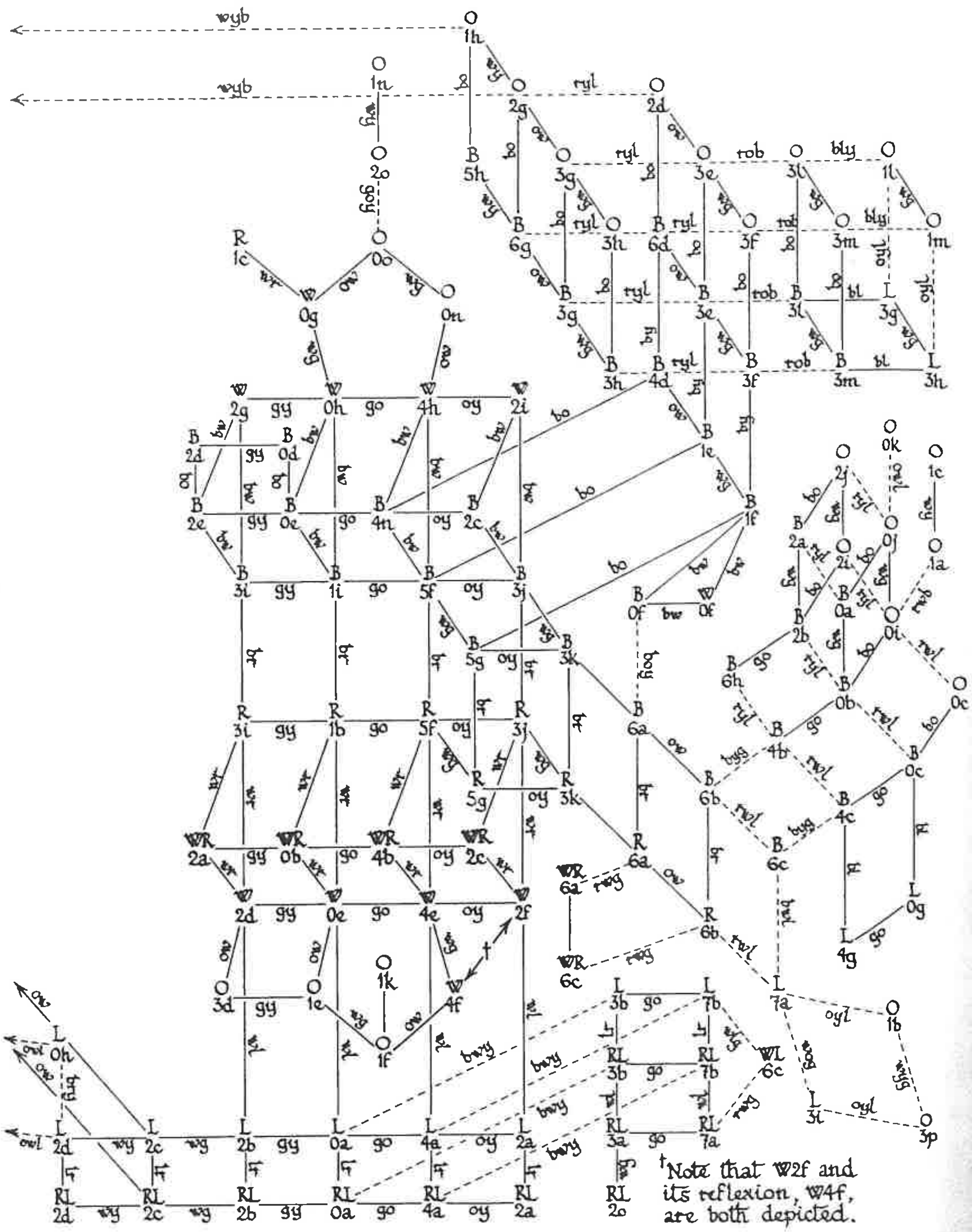
The Soma pieces 1 = W, 2 = Y and 4 = O, while themselves symmetrical, may appear on the surface of the cube in either the *dexter* fashion





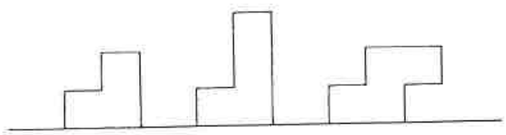
The diamond's gory secrets are seven seas away!

Figure 62. The …



. . . . Somap.

or the *sinister* one



so you can tell which of these pieces are dexter by giving the sum of their numbers, which we call the **dexterity** of the solution. The symbols

DC	DC	DC
na	nb	nc

refer to different solutions having deficient piece D, central piece C and dexterity n, a single capital letter indicating that the same piece is both deficient and central. Thus

RL	RL	RL	RL
5a	5b	5c	5d

are four solutions in which Red is deficient, bLue is central and pieces 1 and 4 are dexter $(1 + 4 = 5)$, while

B	B	B	
6a	6b	6c	...

are solutions in which Black is deficient *and* central while 2 and 4 are dexter.

Along with the solutions in Fig. 62, there are their reflexions whose names are found by interchanging R and L and replacing n by

	$3 - n$,	$6 - n$,	$7 - n$,
in the cases	O central,	W central,	otherwise.

When two solutions are related by changing just two pieces, P and Q, this is indicated by a solid line PQ. Some three-piece changes are indicated by dashed lines in a similar way. So all that's left for you to do is to find a suitable solution which you can locate on the Somap, and this will then lead you to all the others except R7d.

Solutions to the Arithmetico-Geometric Puzzle

Figure 63 shows how we indicate layers in this puzzle by using a or α, according to orientation, for an a-high block, etc. The 21 solutions to Hoffman's puzzle are exhibited in Table 1 in this notation. When, as usual, only the middle layer is shown, another layer is separated from it by a letter S, and the remaining one is the special layer of Fig. 63. The meanings of the other letters in Table 1 are:

- R: reflect the special layer across the dotted diagonal,
- S: swap the two non-special layers,
- S′: swap two adjacent layers in a different direction,
- T: tamper with a $2 \times 2 \times 2$ corner, not involving the special layer,
- T′: tamper with a $2 \times 2 \times 2$ corner, which does involve the special layer.