The Taxman Cometh!

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Acknowledgements

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I also want to thank my colleagues Bob Gardner and Teresa Haynes who took over my classes on Friday so I could come out and play.

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A bit about me...

My favorite games satisfy the following:

(i) A small set of simple rules (i.e., the game has a low threshold).

(ii) Perfect information (i.e., no hidden pieces).

(iii) Opportunity for clever play (i.e., the game has a high ceiling).

(iv) Non-random and quick.

A lot of the above can be used to describe my favorite areas of math (combinatorics, graph theory, group theory, number theory[1]).

[1] Admittedly, my knowledge of number theory ends around the year 1900

About the Taxman

The Taxman game was developed by Diane Resek around 1970. It was published as a computer game by the Minnesota Educational Consortium. However, it can be played easily with pen and paper. The game is easy to explain and quick to play (for small numbers) Hence, it works really well as a contest (player with the largest score wins) in outreach or as part of the math club.

The rules of Taxman

Taxman is a solitaire game. The player starts with a list of all integers 1,...,n, The rules:

(i) The player can remove a number ℓ from the list provided that there is a proper divisor of ℓ on the list.

(ii) (The Tax Rule) The player gets ℓ, while the Taxman collects all remaining proper divisors of ℓ. These numbers are also removed from the list.

(iii) (The "Loose Change" Rule) When no legal moves are possible, the Taxman collects all of the numbers left on the list.

(iv) The player and the Taxman sum up the numbers that they have collected. If the player's score is larger than the Taxman's, then the player wins.

A Sample Game

Let's do a sample game with $n = 12$.

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12$$

Move 1: Player takes 11. Taxman gets 1.

A Sample Game

Let's do a sample game with $n = 12$.

$$\cancel{1} \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad \cancel{11} \quad 12$$

Move 2: Player takes 9. Taxman gets 3.

A Sample Game

Let's do a sample game with $n = 12$.

$$\cancel{1} \quad 2 \quad \cancel{3} \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad \cancel{9} \quad 10 \quad \cancel{11} \quad 12$$

Move 3: Player takes 6. Taxman gets 2.

A Sample Game

Let's do a sample game with $n = 12$.

$$\cancel{1} \quad \cancel{2} \quad \cancel{3} \quad 4 \quad 5 \quad \cancel{6} \quad 7 \quad 8 \quad \cancel{9} \quad 10 \quad \cancel{11} \quad 12$$

Move 4: Player takes 12. Taxman gets 4.

A Sample Game

Let's do a sample game with $n = 12$.

$$\cancel{1} \quad \cancel{2} \quad \cancel{3} \quad \cancel{4} \quad 5 \quad \cancel{6} \quad 7 \quad 8 \quad \cancel{9} \quad 10 \quad \cancel{11} \quad \cancel{12}$$

Move 5: Player takes 10. Taxman gets 5.

A Sample Game

Let's do a sample game with $n = 12$.

$$\cancel{1} \quad \cancel{2} \quad \cancel{3} \quad \cancel{4} \quad \cancel{5} \quad \cancel{6} \quad 7 \quad 8 \quad \cancel{9} \quad \cancel{10} \quad \cancel{11} \quad \cancel{12}$$

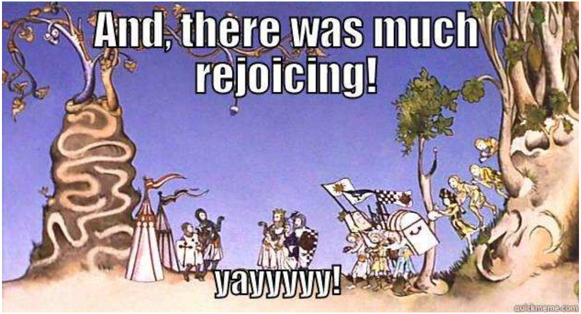
No more legal moves, the Taxman gets the "loose change" of 7 and 8.

Summary of Sample Game

Player has collected 11, 9, 6, 12, and 10. Their score is 48.

The Taxman has collected 1, 3, 2, 4, 5, 7, and 8. The Taxman's score is 30.

The Player Wins!



Beating the Taxman

Hensley [1] and Perlmutter [3] give different asymptotic strategies for beating the Taxman when $n \geq 4$. While these algorithms are quadratic time, both strategies are complicated and involve (somewhat long) look-up tables.

Optimizing the Player's Score

In any case, a more interesting problem is how to *maximize* the Player's score?

"If I knew how to save more on my taxes, then I wouldn't have to be an intergalactic mercenary and get into wacky adventures for money!"

An Optimal Strategy for $n = 12$

In our previous example with $n = 12$, the player scored 48. Is this optimal?
No! As before, the Player takes 11 and 9 on the first two moves. This gives the Taxman 1 and 3.

$$\cancel{1} \quad 2 \quad \cancel{3} \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad \cancel{9} \quad 10 \quad \cancel{11} \quad 12$$

Move 3: Player takes 10. Taxman gets 2 and 5.

An Optimal Strategy for $n = 12$

In our previous example with $n = 12$, the player scored 48. Is this optimal?

$$\cancel{1} \quad \cancel{2} \quad \cancel{3} \quad 4 \quad \cancel{5} \quad 6 \quad 7 \quad 8 \quad \cancel{9} \quad \cancel{10} \quad \cancel{11} \quad 12$$

Move 4: Player takes 8. Taxman gets 4.

An Optimal Strategy for $n = 12$

In our previous example with $n = 12$, the player scored 48. Is this optimal?

$$\cancel{1} \quad \cancel{2} \quad \cancel{3} \quad \cancel{4} \quad \cancel{5} \quad 6 \quad 7 \quad \cancel{8} \quad \cancel{9} \quad \cancel{10} \quad \cancel{11} \quad 12$$

Move 5: Player takes 12. Taxman gets 6.

An Optimal Strategy for $n = 12$

In our previous example with $n = 12$, the player scored 48. Is this optimal?

$$\cancel{1} \quad \cancel{2} \quad \cancel{3} \quad \cancel{4} \quad \cancel{5} \quad \cancel{6} \quad 7 \quad \cancel{8} \quad \cancel{9} \quad \cancel{10} \quad \cancel{11} \quad \cancel{12}$$

No more legal moves. Taxman gets the “loose change” of 7.

Summary of Optimal Game

Player has collected 11, 9, 10, 8, and 12. Their score is 50.

The Taxman has collected 1, 3, 2, 5, 4, 6, and 7. The Taxman's score is 28.

Notice that order matters! Also, the third move of 10 was non-intuitive as it gave the Taxman two divisors.

The Best First Move

Theorem [2] The Player's best first move is to take the largest prime in the list.

Why?

- The Taxman always gets 1 on first turn.
- The Player will get at most one prime. This can only happen on the first turn.
- Bertrand's postulate guarantees a prime between n and $2n$. Hence, there is no multiple of p_{max} on the list. Thus, this choice does not prevent any sequence of nonprime picks.

An Open Question

What is the best *second* move?

“Even I can’t see the best second move!”

The best the Player can possibly do

Suppose that $n = 2t$. Note that the total sum of all of the numbers is $t(2t+1)$. The best that the Player can ever do is to take all of the numbers $t+1$,...,$2t$ and give the Taxman 1,...,t.
This results in a score of $\frac{t(t+1)}{2}$ for the Taxman - roughly 25 percent of the total.
This results in a score of $\frac{t(3t+1)}{2}$ for the Player. At best, the Player can take a bit less than 75 percent of the total.
Note that this bound is achievable - Take $n = 10$. The Player's moves are 7, 9, 6, 8, and 10.
In general, there is more than one prime between $n/2$ and n. Thus, we cannot achieve this bound. This was the case with $n = 12$.

How much is the Taxman guaranteed?

Recall that the Player can take at most one prime. Thus, the Taxman is guaranteed all of the primes on the list, except the largest prime.

The Prime Number Theorem says that the density of primes near n is approximately $1/\ln(n)$. This guarantees that the Taxman is guaranteed roughly $1/ln(n)$ of the total numbers just from the primes [2].

A Greedy Algorithm

A *greedy strategy* is one in which you do the best you can on an individual step, without regard to the consequences in the future. Greedy strategies are easy to describe and provide a bound. However, they may not provide an *optimal strategy*. One greedy strategy for Taxman is to take the largest number available. If we try this with $n = 6$, the player takes 6. The Taxman gets the proper divisors 1, 2, 3. He then gets 4 and 5 as "loose change". The Taxman wins 15 to 6!

Moniot's Greedy Algorithm

Moniot [2] describes a better greedy strategy for the Taxman. Namely, he picks a number ℓ such that it maximizes the difference between ℓ and its remaining proper divisors.

"Only a great fool would reach for what he was given. I'm not a great fool, so I can clearly not choose the wine in front of you. But you must have known I was not a great fool; you would have counted on it, so I can clearly not choose the wine in front of me..."

Greedy Strategy - Example

Let's try $n = 15$.

List	1	2	3	4	5	6	7	8
Tax		1	1	1,2	1	1,2,3	1	1,2,4
Diff		1	2	1	4	0	6	1
List	9	10	11	12	13	14	15	
Tax	1,3	1,2,5	1	1,2,3,4,6	1	1,2,7	1,3,5	
Diff	5	2	10	-4	12	4	6	

The greedy strategy maximizes the difference by taking 13 and giving the Taxman 1. According to Moniot [2] the greedy strategy will take the largest prime for all $n \leq 500000$.

Moniot's Strategy - Example

Let's try $n = 15$.

List	$\cancel{1}$	2	3	4	5	6	7	8
Tax				2		2,3		2,4
Diff				2		1		2
List	9	10	11	12	$\cancel{13}$	14	15	
Tax	3	2,5		2,3,4,6		2,7	3,5	
Diff	6	3		-3		5	7	

The greedy strategy maximizes the difference by taking 15 and giving the Taxman 3 and 5. Since the only divisor of 9 is removed, the Taxman will eventually get 9 as part of the "loose change"!

Moniot's Strategy - Example (Summary)

We now summarize the moves generated by the greedy strategy:

Move	1	2	3	4	5	6	Loose Change
Pick	13	15	10	14	8	12	
Tax	1	3,5	2	7	4	6	9,11

Player's score: 72
Taxman's score: 48

Improved Greedy Strategy

The above example could be improved by taking 9 before 15 as all of the proper divisors of 9 are also proper divisors of 15. Moniot defines such a number as a “freebie” because it can be picked between greed picks without changing the rest of the sequence.

Formally, a number f is a *freebie* on a turn of the greedy strategy if the remaining proper divisors of f are a subset of the proper divisors of the greedy pick AND f is not a divisor of any other number in the greedy sequence.

Improved Greedy Strategy - Summary

A summary of the moves generated by the improved greedy strategy:

Move	1	2	3	4	5	6	7	Loose Change
Pick	13	9	15	10	14	8	12	
Tax	1	3	5	2	7	4	6	11

Player's score: 81
Taxman's score: 39

Improved Greedy May Not Be Optimal

In the appendix to [2], Moniot provides optimal strategies for when all $n \leq 49$, as determined through brute force computation. For most of these, the improved greedy strategy is not optimal. For example:

$n = 13$
Greedy: 13, 9, 6, 12, 10. Total: 50
Optimal: 13, 9, 10, 8, 12. Total: 52

$n = 18$
Greedy: 17, 15, 10, 14, 16, 12, 18. Total: 102
Optimal: 17, 9, 15, 10, 18, 14, 12, 16. Total: 111

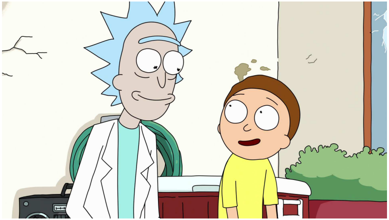
$n = 21$
Greedy: 19, 21, 14, 15, 20, 16, 12, 18. Total: 135
Optimal: 19, 9, 21, 15, 14, 18, 12, 20, 16. Total: 144

Why is Taxman hard?

- Order of the picks is important! I estimate that there are $O(\sqrt{n!})$ possible games.
- Finding the optimal solution seems to be NP-complete.
- Combines addition and multiplication. See for example Goldbach Conjecture, Fermat's Last Theorem, Collatz Conjecture,...
- The "loose change" can provide a big boost to the Taxman's score. In our initial example with $n = 12$, half of the Taxman's score came from the loose change.

More Open Questions

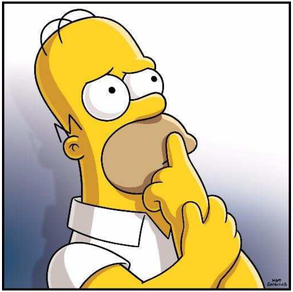
What if the Player MUST pick certain numbers?
What if the Player CANNOT pick certain numbers?
What if the Taxman MUST receive certain numbers?



"Gee Rick, that sounds a Garden of Eden type problem."

More Open Problems (Part 2)

Consider a version in which the Player tries to play as badly as possible. In other words, the Player is trying to *minimize* their own score. Equivalently, they are trying to *maximize* the Taxman's score.



"Sounds like the fool's solitaire problem of Beeler and Rodriguez!"

More Open Problems (Part 3)

Consider a two-player version. The players alternate taking removing numbers from the list. A number ℓ can be removed from the list, provided that it has at least one proper divisor on the list. All remaining proper divisors of ℓ are then given to their opponent. What is the optimal strategy for each player?

"Hope that the players are smarter than us."

More Open Problems (Part 4)

Suppose that the Player plays "randomly."

- What is the Player's expected score?
- What is the expected proportion of games that the Player will win with this strategy?
- What distribution (uniform, left-skewed, etc.) tends to give the best outcome?

"Do I really look like a guy with a plan?"

More Open Problems (Part 5)

We can also consider alternative strategies and see how they compare against Moniot's improved greedy strategy. Possibilities:

- The Player tries to minimize how many numbers they give away .
- The Player tries to minimize how much "loose change" they give the Taxman. Note that this is equivalent to the cooperative two-player game.

"With different strategies, we can boldly go where no one has gone before."

Questions?

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