5.1 Estimating with Confidence

**Note.** Recall the facts about the sampling distribution of $\bar{x}$:

- $\bar{x}$ has a normal distribution.
- The mean of this normal sampling distribution is the same as the unknown population mean.
- The standard deviation of $\bar{x}$ for an SRS of size $n$ is $\sigma/\sqrt{n}$ where $\sigma$ is the standard deviation of the population. (It is not realistic to assume that we know $\sigma$. We will see in the next chapter how to proceed when $\sigma$ is not known.)

### Statistical Confidence

**Definition.** A *confidence interval* is of the form

$$\text{estimate} \pm \text{margin of error}.$$ 

The *margin of error* shows how accurate we believe our guess is, based on the variability of the estimate.

**Example 5.2.** The NAEP survey includes a short test of quantitative skills, covering mainly basic arithmetic and the ability to apply it to realistic problems. Scores on the test range from 0 to 500. For example, a person who scores 233 can add the amounts of two checks appearing on a bank deposit slip; someone scoring 325 can determine the price of a meal from a menu; a person scoring 375 can transform a price in
cents per ounce into dollars per pound. In a recent year, 840 men 21 to 25 years of age were in the NAEP sample. Their mean quantitative score was $\bar{\pi} = 272$. These 840 men are an SRS from the population of all young men. On the basis of this sample, what can we say about the mean score $\mu$ in the population of all 9.5 million young men of these ages?

**Solution.** The standard deviation of $\bar{x}$ is $\sigma/\sqrt{n} = 60/\sqrt{840} = 2.1$. Figure 5.1 (and TM-80) gives the sampling distribution for $\bar{\pi}$. If we want a 95% confidence interval for $\mu$, we should go two standard deviations from the sample mean (recall the 68-95-99.7 rule). Since $\bar{\pi} = 272$, and the sample standard of deviation is 2.1, we set the margin of error equal to $2 \times 2.1 = 4.2$ and so the confidence interval is from $272 - 4.2 = 267.8$ to $272 + 4.2 = 276.2$. Therefore we can say that we are 95% confident that the population mean $\mu$ lies between 267.8 and 276.2.

**Confidence Intervals**

**Note.** Any confidence interval has two parts: an interval computed from the data and a confidence level giving the probability that the method produces an interval that covers the parameter.

**Definition.** A level $C$ confidence interval for a parameter is an interval computed from sample data by a method that has probability $C$ of producing an interval containing the true value of the parameter.
Example 5.3. To find an 80% confidence interval, we must catch the central 80% we leave out 20% or 10% in each tail. So $z^*$ is the point with area 0.1 to its right (and 0.9 to its left) under the standard normal curve. Search the body of Table A (TM-139, TM-140) to find the point with area 0.9 to its left. The closest entry is $z^* = 1.28$. There is area 0.8 under the standard normal curve between -1.28 and 1.28. Figure 5.4 (TM-83) shows how $z^*$ is related to areas under the curve.

Note. Figure 5.5 (and TM-84) shows the general situation for any confidence level $C$. If we catch the central area $C$, the leftover tail area is $1 - C$, or $(1 - C)/2$ on each side. You can find $Z^*$ for any $C$ by searching Table A (TM-139, TM-140). Here are the results for the most common confidence levels:

<table>
<thead>
<tr>
<th>Confidence Level</th>
<th>Tail Area</th>
<th>$z^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>90%</td>
<td>.05</td>
<td>1.645</td>
</tr>
<tr>
<td>95%</td>
<td>.025</td>
<td>1.960</td>
</tr>
<tr>
<td>99%</td>
<td>.005</td>
<td>2.576</td>
</tr>
</tbody>
</table>

Definition. The number $z^*$ with probability $p$ lying to its right under the standard normal curve is called the upper $p$ critical value of the standard normal distribution.

Definition. Draw an SRS of size $n$ from a population having unknown mean $\mu$ and known standard deviation $\sigma$. A level $C$ confidence interval for $\mu$ is

$$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}.$$
Here \( z^* \) is the upper \((1 - C)/2\) critical value for the standard normal distribution, found in Table C (TM-142). This interval for the standard normal distribution is normal and is approximately correct for large \( n \) in other cases.

**Example 5.4.** A manufacturer of pharmaceutical products analyzes a specimen from each batch of a product to verify the concentration of the active ingredient. The chemical analysis is not perfectly precise. Repeated measurements on the same specimen give slightly different results. The results of repeated measurements follow a normal distribution quite closely. The analysis procedure has no bias, so the mean \( \mu \) of the population of all measurements is the true concentration in the specimen. The standard deviation of this distribution is known to be \( \sigma = .0068 \) grams per liter. The laboratory analyzes each specimen three times and reports the mean result. Three analyses of one specimen give concentrations

\[
0.8403 \quad 0.8363 \quad 0.8447.
\]

We want a 99% confidence interval for the true concentration \( \mu \). The sample mean of the three readings is

\[
\bar{x} = \frac{.8403 + .8363 + .8447}{3} = .8404.
\]

For 99% confidence, we see from Table C (TM-142) that \( z^* = 2.576 \). A 99% confidence interval for \( \mu \) is therefore

\[
\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}} = .8404 \pm \frac{.0068}{\sqrt{3}} = .8404 \pm .0101 = (.8303, .8505).
\]

We are 99% confident that the true concentration lies between 0.8303 and 0.8505 grams per liter.
Note. For a confidence interval, the margin of error is $z^*\sigma/\sqrt{n}$. The expression has $z^*$ and $\sigma$ in the numerator and $\sqrt{n}$ in the denominator. So the margin of error gets smaller when:

- $z^*$ gets smaller.
- $\sigma$ gets smaller.
- $n$ gets larger.

Choosing the Sample Size

Note. The confidence interval for a population mean will have a specified margin of error $m$ when the sample size is 

$$n = \left(\frac{z^*\sigma}{m}\right)^2.$$

Example 5.6. Management asks the laboratory of Example 5.4 to produce results accurate to within $\pm 0.005$ with 95% confidence. How many measurements must be averaged to comply this request?

Solution. The desired margin of error is $m = 0.005$. For 95% confidence, Table C (TM-142) gives $z^* = 1.960$. We know that $\sigma = 0.0068$. Therefore:

$$n = \left(\frac{z^*\sigma}{m}\right) = \left(\frac{1.96 \times 0.0068}{0.005}\right)^2 = 7.1.$$ 

Because 7 measurements will give a slightly larger margin of error than desired, and 8 measurements a slightly smaller margin of error, the lab
must take 8 measurements on each specimen to meet management’s demand.

Some Cautions

**Note.** Some warnings:

- The data must be an SRS from the population.
- The formula is not correct for probability sampling designs more complex than an SRS.
- There is no correct method for inference from data haphazardly collected with bias of unknown size.
- Because \( \bar{x} \) is strongly influenced by a few extreme observations, outliers can have a large effect on the confidence interval.
- If the sample size is small and the population is not normal, the true confidence level will be different from the value \( C \) used in computing the interval.
- You must know the standard deviation \( \sigma \) of the population.