Chapter 12. General Rules of Probability

Independence and the Multiplication Rule

Note. It is sometimes helpful when dealing with multiple outcomes of an experiment, to draw a **Venn diagram** for the experiment. Suppose an experiment has a sample space S with possible outcomes A and B. In addition, suppose that both outcomes A and B can occur together. Then a Venn diagram for the experiment is something like the following.
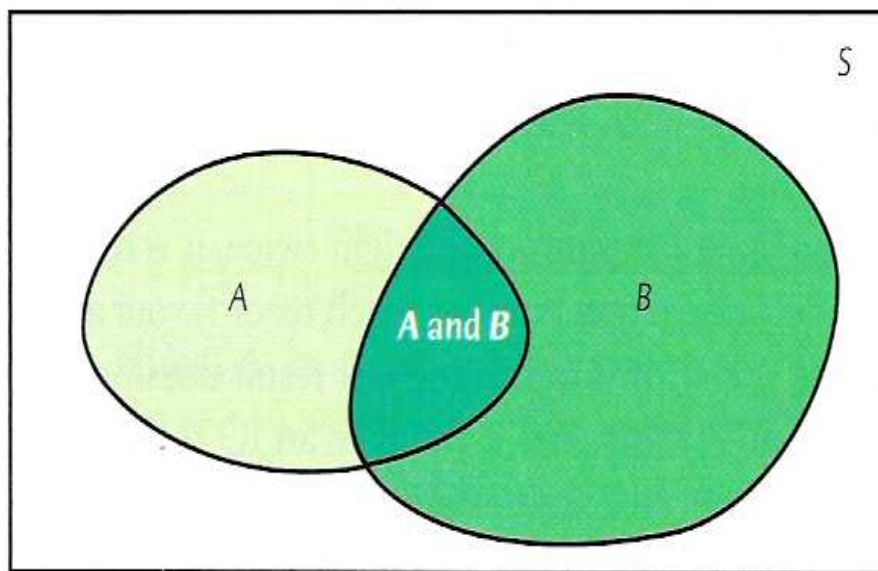

Figure 12.2 Page 303

Note. An example of such an experiment is to pick a card from a standard deck of 52 (no jokers). Event A could represent the outcome that the card is red (i.e., a heart or diamond) and event B could represent the outcome that the card is a face card (i.e., a jack, queen, or king). Then events A and B together is the outcome that the card is a red face card.

Definition. Two events A and B are **independent** if knowing that one occurs does not change the probability that the other occurs. If A and B are independent,

$$P(A \text{ and } B) = P(A)P(B).$$

Note. A common mistake in the misapplication of probability is to assume independence of events which are not independent. The text describes such a situation on page 305. Another common misuse is in "creation science" or "intelligent design" in which probabilities of chains of amino acids in proteins are calculated with an assumption on independence, which is not the case in nature. For example, a quick Google search (2/7/2009) reveals this website:

http://www.geocities.com/athens/aegean/8830/mathproofcreat.html

Notice the section "Mathematical Proof Of The Intelligent Design Of Proteins." The conclusion that these events could not happen "at random" or "by chance" is correct—the mechanism governing these event is natural selection (not a random process at all).

Example. A probability argument can be made against the widespread testing of a population for a certain medical condition (HIV, drug use, or even pregnancy). Consider Example 12.3 on page 305.

The General Addition Rule

Definition. For any two events A and B,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

See Figure 12.2 again for the justification of this.

Note. If events A and B are disjoint (i.e., they cannot both occur at the same time), then $P(A \text{ and } B) = 0$. Such events are sometimes called "mutually exclusive." If A and B are mutually exclusive, then things are simplified and $P(A \text{ or } B) = P(A) + P(B)$.

Example S.12.1. Adding Stooges.

If we break the 190 Three Stooges films into two categories, one for the director and one the the role of the third stooge, we get the following:

3rd Stooge\Director	Del Lord	Jules White	Ed Bernds	Other	Total
Curly	38	33	5	21	97
Shemp	1	55	20	1	77
Joe	0	16	0	0	16
Total	39	104	25	22	190

Denote events as: C represents the event of Curly as the third Stooge, S represents Shemp as the third stooge, J represents

Joe as the third stooge, L represents the event Del Lord is the director, W represents Jules White is the director, B represents Ed Bernds is the director, and O represents a director other than Lord, White, or Bernds. Suppose the experiment of choosing a Three Stooges film at random is performed. Answer the following questions.

1. What is $P(C)$, $P(S)$, $P(J)$, $P(L)$, $P(W)$, $P(B)$, and $P(O)$?

2. In words, what is the event "S or L"? What is $P(S$ or $L)$?

3. What is $P(C$ or $W)$?

4. Are C and B disjoint events?

5. Give an example of two disjoint events.

Conditional Probability

Note. Sometimes the occurrence of one event A can affect the probability of another event B. In this case, we discuss a "conditional probability" $P(B \mid A)$, read "the probability of event B given that event A has occurred." For example, if we roll a six-sided die and want to know the probability of rolling a 2 (call this event B), then $P(B) = 1/6$. However, if we know that an even number has been rolled (call this event A), then $P(B \mid A) = 1/3$.

Definition. When $P(A) > 0$, the **conditional probability** of B given A is

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}.$$

Example S.12.2. Conditional Stooges.

Use the third stooge/director information from Example S.12.1 to answer the following:

1. $P(S \text{ and } L) =$? $P(S \mid L) =$?

2. $P(C \text{ and } W) =$? $P(C \mid W) =$?

3. What is $P(W \mid C)$? Is this the same as $P(C \mid W)$?

Example. What is the probability that three people chosen at random all have different birthdays? What is the probability that four people chosen at random all have different birthdays? What about five people? In a group of five people chosen at random, what is the probability that at least two of the people have the same birthday (notice that this is the complement of the event where all have different birthdays)?

Example. The Birthday Problem.

In a group the size of this class (say size n), we find that the probability P that at least two of us share a birthday is:

n	2	3	4	5	10	15	20	21	22	23	24
P	0.0027	0.0082	0.0164	0.0271	0.1169	0.2529	0.4110	0.4437	0.4757	0.5073	0.5383

n	30	31	32	33	34	35	40	45	50	100	365
P	0.7063	0.7305	0.7533	0.7750	0.7953	0.8144	0.8912	0.9410	0.9704	0.9999997	1.000

Let's go around the room and see if we get the expected outcome.

The General Multiplication Rule

Definition. The probability of both of two events A and B happen together can be found by

$$P(A \text{ and } B) = P(A)P(B \mid A).$$

Here $P(B \mid A)$ is the conditional probability that B occurs, given the information that A occurs.

Example S.12.3. More Conditional Stooges.

The data from Example S.12.1 implies that $P(W \mid S) = 55/77$ and $P(S) = 77/190$. Use this information to calculate $P(W \text{ and } S)$. Is this consistent with the table of information in Example S.12.1?

Independence

Definition. Two events A and B that both have positive probability are **independent** if $P(A \mid B) = P(B)$.

Example S.12.4. Dependent Stooges.

Are any of the "third stooge" events C, S, and J independent of the "director" events L, W, B, and O?

Partial Solution. This requires us to compute the probabilities for each of these 7 events, plus 12 conditional probabilities. We will find that the answer is "no!" in each case. For example, we have from Example S.12.1 that $P(W) = 104/190$ and from Example S.12.3 that $P(W \mid S) = 55/77$. From this, it follows that

$$P(W \mid S) = 77/190 \neq P(W) = 104/190.$$

Therefore these events are not independent.

Tree Diagrams

Note. We illustrate the use of tree diagrams with an example.

Example. Exercise 12.52 page 323.

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