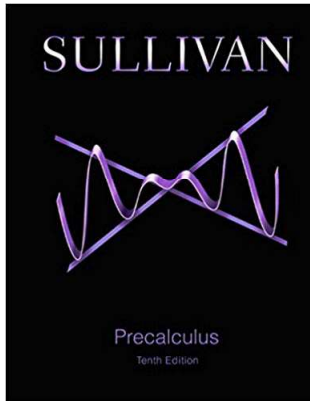


Precalculus 1 (Algebra)

Chapter 1. Graphs

1.1. The Distance and Midpoint Formulas—Exercises, Examples, Proofs



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Precalculus 1 (Algebra)

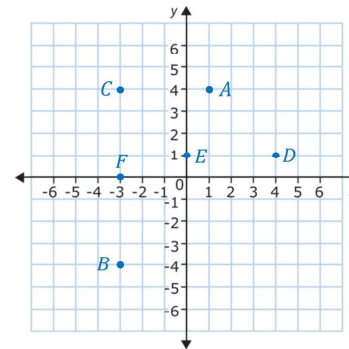
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Page 7 Number 16

Page 7 Number 16. Plot each point in the xy -plane. Tell in which quadrant or on what coordinate axis each point lies:

- (a) $A = (1, 4)$ (d) $D = (4, 1)$
 (b) $B = (-3, -4)$ (e) $E = (0, 1)$
 (c) $C = (-3, 4)$ (f) $F = (-3, 0)$

Solution. We have the Cartesian plane:



The points are as graphed. They appear in the following locations:

Point	Location
A	Quadrant I
B	Quadrant III
C	Quadrant II
D	Quadrant I
E	y -axis
F	x -axis

□

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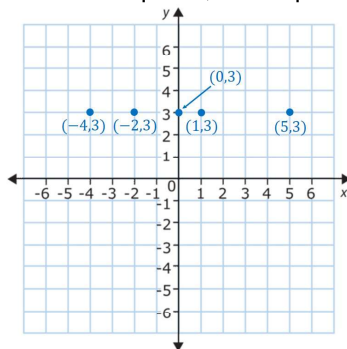
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Page 7 Number 18

Page 7 Number 18. Plot the points $(0, 3)$, $(1, 3)$, $(-2, 3)$, $(5, 3)$, and $(-4, 3)$. Describe the set of all points of the form $(x, 3)$ where x is any real number.

Solution. We have the Cartesian plane; the 5 points are as graphed.



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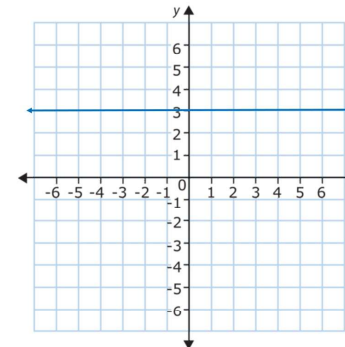
Precalculus 1 (Algebra)

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Page 7 Number 18 (continued)

Page 7 Number 18. Plot the points $(0, 3)$, $(1, 3)$, $(-2, 3)$, $(5, 3)$, and $(-4, 3)$. Describe the set of all points of the form $(x, 3)$ where x is any real number.

Solution (continued). The points of the form $(x, 3)$ yield a horizontal line:



□

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Precalculus 1 (Algebra)

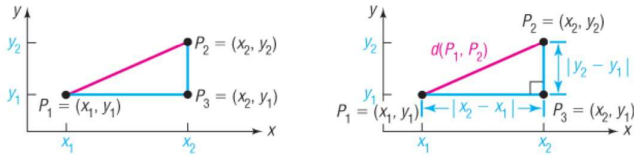
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Theorem 1.1.A

Theorem 1.1.A. The Distance Formula. The distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, denoted $d(P_1, P_2)$, is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Proof. First, assume that the line joining P_1 and P_2 is neither horizontal nor vertical:



The coordinates of point P_3 are (x_2, y_1) . The horizontal distance from P_1 to P_3 is the absolute value of the difference of the x coordinates (because this is how distance is measured on the real line), $|x_2 - x_1|$. The vertical distance from P_3 to P_2 is the absolute value of the difference of the y -coordinates $|y_2 - y_1|$. Notice that $P_1P_2P_3$ is a right triangle.

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Theorem 1.1.A (continued 1)

Theorem 1.1.A. The Distance Formula. The distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, denoted $d(P_1, P_2)$, is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Proof (continued). The hypotenuse of the right triangle has length $d(P_1, P_2)$, so by the Pythagorean Theorem (see Appendix A.2. Geometry Essentials)

$$[d(P_1, P_2)]^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

and so (since distances are always positive)

$$\sqrt{d(P_1, P_2)^2} = |d(P_1, P_2)| = d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

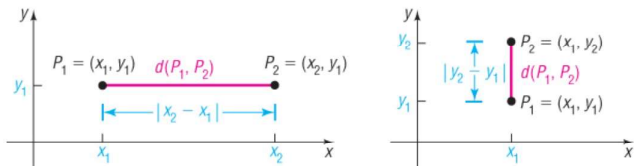
as claimed.

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Theorem 1.1.A (continued 2)

Proof (continued). Second, if the line joining P_1 and P_2 is horizontal, then the y -coordinate of P_1 equals the y -coordinate of P_2 ; that is, $y_1 = y_2$. In this case $d(P_1, P_2)$ is simply $|x_2 - x_1|$ and the distance formula still holds since

$$d(P_1, P_2) = |x_2 - x_1| = \sqrt{(x_2 - x_1)^2 + 0} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



If the line joining P_1 and P_2 is vertical, then the x -coordinate of P_1 equals the x -coordinate of P_2 ; that is, $x_1 = x_2$. In this case $d(P_1, P_2)$ is simply $|y_2 - y_1|$ and the distance formula still holds since

$$d(P_1, P_2) = |y_2 - y_1| = \sqrt{0 + (y_2 - y_1)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \quad \square$$

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Page 7 Number 26

Page 7 Number 26. Find the distance between the points $P_1 = (2, -3)$ and $P_2 = (4, 2)$.

Solution. We use the distance formula

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where we have $P_1 = (x_1, y_1) = (2, -3)$ and $P_2 = (x_2, y_2) = (4, 2)$. So we have $x_1 = 2$, $y_1 = -3$, $x_2 = 4$, and $y_2 = 2$. Hence the distance is

$$\begin{aligned} d(P_1, P_2) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(4 - 2)^2 + (2 - (-3))^2} \\ &= \sqrt{2^2 + (5)^2} = \sqrt{4 + 25} = \sqrt{29}. \end{aligned}$$

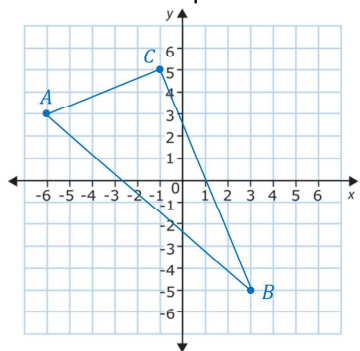
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Page 7 Number 34

Page 7 Number 34. Plot each point $A = (-6, 3)$, $B = (3, -5)$, and $C = (-1, 5)$, and form the triangle ABC . Show that the triangle is a right triangle and give its area.

Solution. The points and triangle ABC are:



The lengths of the edges of ABC are given by the distances:

$$d(A, B) = \sqrt{(3 - (-6))^2 + (-5 - 3)^2} \\ = \sqrt{81 + 64} = \sqrt{145},$$

$$d(A, C) = \sqrt{(-1 - (-6))^2 + (5 - 3)^2} \\ = \sqrt{25 + 4} = \sqrt{29}, \text{ and}$$

$$d(B, C) = \sqrt{(-1 - (3))^2 + (5 - (-5))^2} \\ = \sqrt{16 + 100} = \sqrt{116}.$$

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Page 7 Number 34 (continued)

Page 7 Number 34. Plot of the points each point $A = (-6, 3)$, $B = (3, -5)$, and $C = (-1, 5)$, and form the triangle ABC . Show that the triangle is a right triangle and give its area.

Solution (continued). By the Converse of the Pythagorean Theorem (see Appendix A.2. Geometry Essentials), ABC is a right triangle if the square of the length of one side is the sum of the squares of the lengths of the other two sides. We have $(\sqrt{145})^2 = (\sqrt{29})^2 + \sqrt{116}^2$ (since $145 = 29 + 116$) and so $d(A, B)^2 = d(A, C)^2 + d(B, C)^2$. Therefore, ABC is a right triangle with the right angle at vertex C and with side AB as the hypotenuse.

The area of a triangle is $1/2$ the base times the height. With side AC as the base and side BC as the height, we then have the area $A = (1/2)d(A, C)d(B, C) = (1/2)\sqrt{29}\sqrt{116} = 58/2 = \boxed{29}$. \square

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Theorem 1.1.B

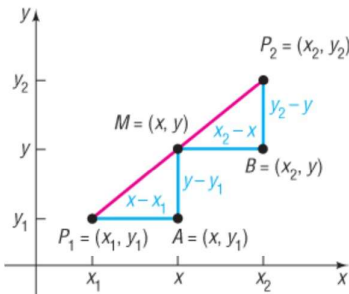
Theorem 1.1.B. The Midpoint Formula. The midpoint $M = (x, y)$ of the line segment from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$ is

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

Proof. Consider the triangles P_1AM and MBP_2 :

By classical Euclidean geometry, triangles P_1AM and MBP_2 are congruent (by angle-side-angle). So corresponding sides of the triangles are equal and

$$\begin{aligned} x - x_1 &= x_2 - x & y - y_1 &= y_2 - y \\ 2x &= x_1 + x_2 & 2y &= y_1 + y_2 \\ x &= \frac{x_1 + x_2}{2} & y &= \frac{y_1 + y_2}{2} \end{aligned}$$

 \square 

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Page 7 Number 40

Page 7 Number 40. Find the midpoint of the line segment joining the points $P_1 = (2, -3)$ and $P_2 = (4, 2)$.

Solution. We use the midpoint formula

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

where $P_1 = (x_1, y_1) = (2, -3)$ and $P_2 = (x_2, y_2) = (4, 2)$. So we have $x_1 = 2$, $y_1 = -3$, $x_2 = 4$, and $y_2 = 2$. So the midpoint is

$$\begin{aligned} M = (x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left(\frac{2 + 4}{2}, \frac{-3 + 2}{2} \right) \\ &= \left(\frac{6}{2}, \frac{-1}{2} \right) = \boxed{(3, -1/2)}. \end{aligned}$$

 \square

()

Page 7 Number 44

Page 7 Number 44. Find the midpoint of the line segment joining the points $P_1 = (a, a)$ and $P_2 = (0, 0)$.

Solution. We use the midpoint formula

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

where $P_1 = (x_1, y_1) = (a, a)$ and $P_2 = (x_2, y_2) = (0, 0)$. So we have $x_1 = a$, $y_1 = a$, $x_2 = 0$, and $y_2 = 0$. So the midpoint is

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left(\frac{a + 0}{2}, \frac{a + 0}{2} \right) = \left(\frac{a}{2}, \frac{a}{2} \right).$$

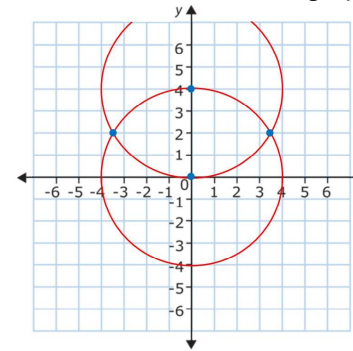
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Page 8 Number 56

Page 8 Number 56. An *equilateral triangle* is one in which all three sides are of equal length. If two vertices of an equilateral triangle are $(0, 4)$ and $(0, 0)$, find the third vertex. How many of these triangles are possible?

Solution. Consider the graph of the two points:



For an equilateral triangle, we need a third point that is also a distance of 4 from both given points. So we draw circles of radius 4 centered at each point.

We see that there are two possible choices for the third point. We now find the coordinates of these two points.

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Page 8 Number 56 (continued)

Page 8 Number 56. An *equilateral triangle* is one in which all three sides are of equal length. If two vertices of an equilateral triangle are $(0, 4)$ and $(0, 0)$, find the third vertex. How many of these triangles are possible?

Solution (continued). Let (x, y) be a point distance 4 from both $(0, 4)$ and $(0, 0)$. Then from the distance formula,
 $4 = \sqrt{(x - 0)^2 + (y - 4)^2} = \sqrt{(x - 0)^2 + (y - 0)^2}$ or, squaring,

$$16 = x^2 + (y - 4)^2 = x^2 + y^2. \quad (*)$$

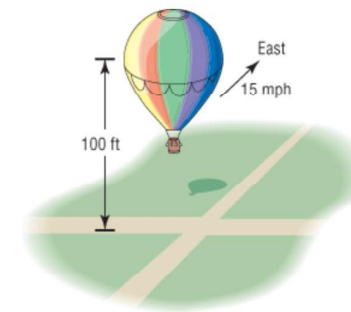
So we need $x^2 + y^2 - 8y + 16 = x^2 + y^2$ or $-8y + 16 = 0$ or $y = 2$. From $(*)$ with $y = 2$, we need (say) $16 = x^2 + (2)^2$ or $x^2 = 12$ or $\sqrt{x^2} = \sqrt{12}$ or $|x| = \sqrt{12} = 2\sqrt{3}$. So we must have $x = \pm 2\sqrt{3}$. The two points which make an equilateral triangle are

$$(x, y) = (2\sqrt{3}, 2) \text{ and } (x, y) = (-2\sqrt{3}, 2). \quad \square$$

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Page 8 Number 68

Page 8 Number 68. A hot-air balloon, headed due east at an average speed of 15 miles per hour and at a constant altitude of 100 feet, passes over an intersection (see the figure). Find an expression for the distance d (measured in feet) from the balloon to the intersection t seconds later. HINT: 1 mile is 5280 feet and 1 hour is 3600 seconds.



Solution. We need to use the distance formula: $(\text{distance}) = (\text{rate})(\text{time})$.

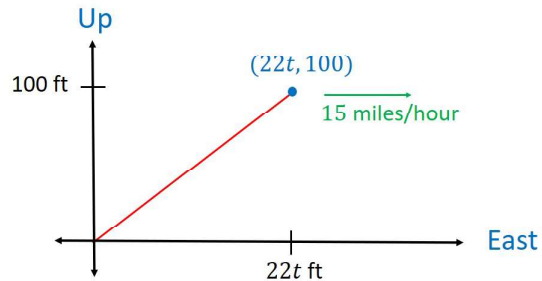
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Page 8 Number 68 (continued 1)

Solution (continued). The horizontal distance traveled after t seconds is

$$\begin{aligned} x &= (15 \text{ miles/hour}) \left(\frac{5280 \text{ feet}}{1 \text{ mile}} \right) (t \text{ seconds}) \left(\frac{1 \text{ hour}}{3600 \text{ seconds}} \right) \\ &= \frac{(15)(5280)t}{3600} \text{ feet} = 22t \text{ feet.} \end{aligned}$$

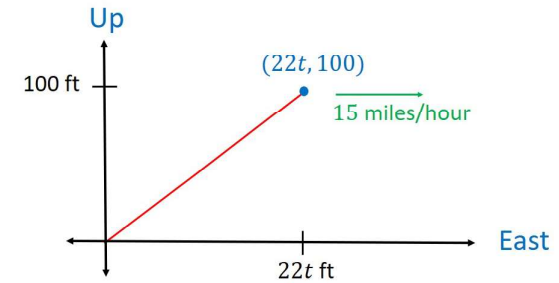
So at time t seconds we have:



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Page 8 Number 68 (continued 2)

Solution (continued).



So with the origin of the above coordinate system at the intersection, the distance from the intersection at $(0, 0)$ to the balloon at $(22t, 100)$ (with all distance units in feet) is

$$\sqrt{(22t - 0)^2 + (100 - 0)^2} = \boxed{\sqrt{484t^2 + 10000} = d}.$$

□

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