Precalculus 1 (Algebra)

Chapter 1. Graphs

1.1. The Distance and Midpoint Formulas—Exercises, Examples, Proofs

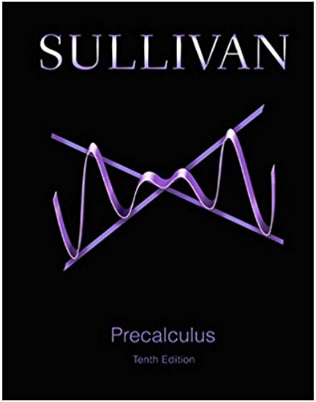


Table of contents

Page 7 Number 16

Page 7 Number 16. Plot each point in the *xy*-plane. Tell in which quadrant or on what coordinate axis each point lies:

(a) $A = (1, 4)$ (d) $D = (4, 1)$
(b) $B = (-3, -4)$ (e) $E = (0, 1)$
(c) $C = (-3, 4)$ (f) $F = (-3, 0)$

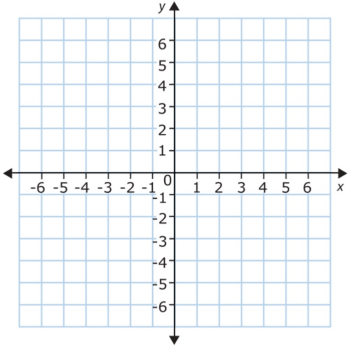
Solution. We have the Cartesian plane:

Page 7 Number 16

Page 7 Number 16. Plot each point in the *xy*-plane. Tell in which quadrant or on what coordinate axis each point lies:

(a) $A = (1, 4)$ (d) $D = (4, 1)$
(b) $B = (-3, -4)$ (e) $E = (0, 1)$
(c) $C = (-3, 4)$ (f) $F = (-3, 0)$

Solution. We have the Cartesian plane:

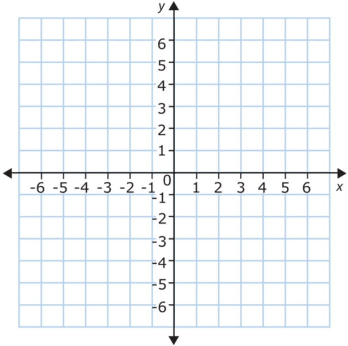


Page 7 Number 16

Page 7 Number 16. Plot each point in the *xy*-plane. Tell in which quadrant or on what coordinate axis each point lies:

(a) $A = (1, 4)$ (d) $D = (4, 1)$
(b) $B = (-3, -4)$ (e) $E = (0, 1)$
(c) $C = (-3, 4)$ (f) $F = (-3, 0)$

Solution. We have the Cartesian plane:

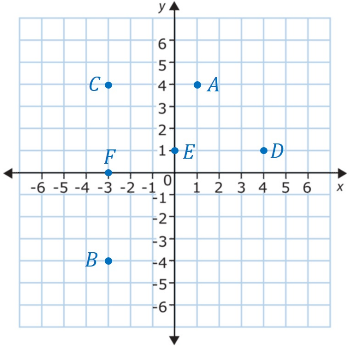


Page 7 Number 16

Page 7 Number 16. Plot each point in the *xy*-plane. Tell in which quadrant or on what coordinate axis each point lies:

(a) $A = (1, 4)$ (d) $D = (4, 1)$
(b) $B = (-3, -4)$ (e) $E = (0, 1)$
(c) $C = (-3, 4)$ (f) $F = (-3, 0)$

Solution. We have the Cartesian plane:



The points are as graphed. They appear in the following locations:

Point	Location
A	Quadrant I
B	Quadrant III
C	Quadrant II
D	Quadrant I
E	y-axis
F	x-axis

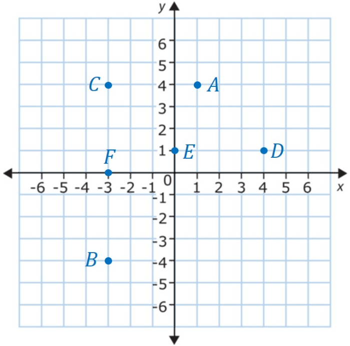
□

Page 7 Number 16

Page 7 Number 16. Plot each point in the *xy*-plane. Tell in which quadrant or on what coordinate axis each point lies:

(a) $A = (1, 4)$ (d) $D = (4, 1)$
(b) $B = (-3, -4)$ (e) $E = (0, 1)$
(c) $C = (-3, 4)$ (f) $F = (-3, 0)$

Solution. We have the Cartesian plane:



The points are as graphed. They appear in the following locations:

Point	Location
A	Quadrant I
B	Quadrant III
C	Quadrant II
D	Quadrant I
E	y-axis
F	x-axis

□

Page 7 Number 18

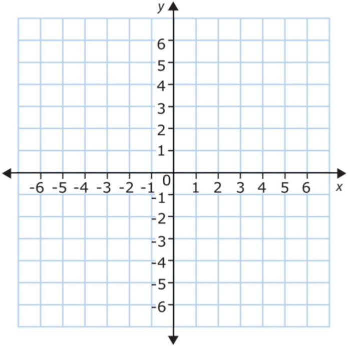
Page 7 Number 18. Plot the points $(0, 3)$, $(1, 3)$, $(-2, 3)$, $(5, 3)$, and $(-4, 3)$. Describe the set of all points of the form $(x, 3)$ where x is any real number.

Solution. We have the Cartesian plane:

Page 7 Number 18

Page 7 Number 18. Plot the points $(0, 3)$, $(1, 3)$, $(-2, 3)$, $(5, 3)$, and $(-4, 3)$. Describe the set of all points of the form $(x, 3)$ where x is any real number.

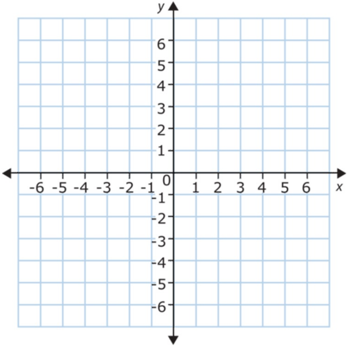
Solution. We have the Cartesian plane:



Page 7 Number 18

Page 7 Number 18. Plot the points $(0,3)$, $(1,3)$, $(-2,3)$, $(5,3)$, and $(-4,3)$. Describe the set of all points of the form $(x,3)$ where x is any real number.

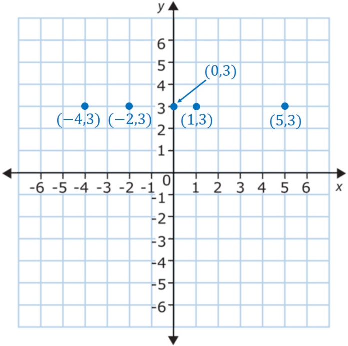
Solution. We have the Cartesian plane:



Page 7 Number 18

Page 7 Number 18. Plot the points $(0, 3)$, $(1, 3)$, $(-2, 3)$, $(5, 3)$, and $(-4, 3)$. Describe the set of all points of the form $(x, 3)$ where x is any real number.

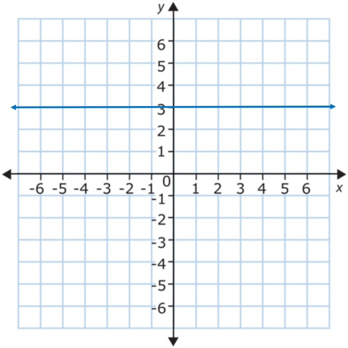
Solution. We have the Cartesian plane; the 5 points are as graphed.



Page 7 Number 18 (continued)

Page 7 Number 18. Plot the points $(0, 3)$, $(1, 3)$, $(-2, 3)$, $(5, 3)$, and $(-4, 3)$. Describe the set of all points of the form $(x, 3)$ where x is any real number.

Solution (continued). The points of the form $(x, 3)$ yield a horizontal line:



Theorem 1.1.A

Theorem 1.1.A. The Distance Formula. The distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, denoted $d(P_1, P_2)$, is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

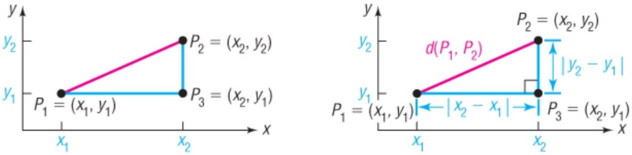
Proof. First, assume that the line joining P_1 and P_2 is neither horizontal nor vertical:

Theorem 1.1.A

Theorem 1.1.A. The Distance Formula. The distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, denoted $d(P_1, P_2)$, is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Proof. First, assume that the line joining P_1 and P_2 is neither horizontal nor vertical:



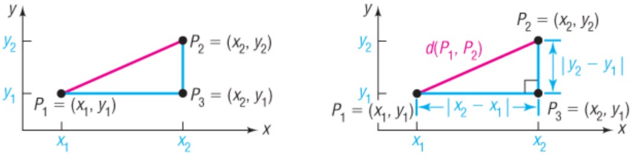
The coordinates of point P_3 are (x_2, y_1). The horizontal distance from P_1 to P_3 is the absolute value of the difference of the x coordinates (because this is how distance is measured on the real line), $|x_2 - x_1|$. The vertical distance from P_3 to P_2 is the absolute value of the difference of the y-coordinates $|y_2 - y_1|$. Notice that $P_1P_2P_3$ is a right triangle.

Theorem 1.1.A

Theorem 1.1.A. The Distance Formula. The distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, denoted $d(P_1, P_2)$, is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Proof. First, assume that the line joining P_1 and P_2 is neither horizontal nor vertical:



The coordinates of point P_3 are (x_2, y_1). The horizontal distance from P_1 to P_3 is the absolute value of the difference of the x coordinates (because this is how distance is measured on the real line), $|x_2 - x_1|$. The vertical distance from P_3 to P_2 is the absolute value of the difference of the y-coordinates $|y_2 - y_1|$. Notice that $P_1P_2P_3$ is a right triangle.

Theorem 1.1.A (continued 1)

Theorem 1.1.A. The Distance Formula. The distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, denoted $d(P_1, P_2)$, is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Proof (continued). The hypotenuse of the right triangle has length $d(P_1, P_2)$, so by the Pythagorean Theorem (see Appendix A.2. Geometry Essentials)

$$[d(P_1, P_2)]^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

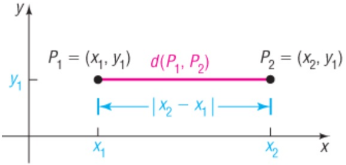
and so (since distances are always positive)

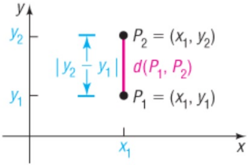
$$\sqrt{d(P_1, P_2)^2} = |d(P_1, P_2)| = d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

as claimed.

Theorem 1.1.A (continued 2)

Proof (continued). Second, if the line joining P_1 and P_2 is horizontal, then the y-coordinate of P_1 equals the y-coordinate of P_2; that is, $y_1 = y_2$. In this case $d(P_1, P_2)$ is simply $|x_2 - x_1|$ and the the distance formula still holds since
$d(P_1, P_2) = |x_2 - x_1| = \sqrt{(x_2 - x_1)^2 + 0} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

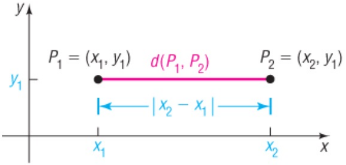


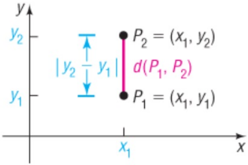


If the line joining P_1 and P_2 is vertical, then the x-coordinate of P_1 equals the x-coordinate of P_2; that is, $x_1 = x_2$. In this case $d(P_1, P_2)$ is simply $|y_2 - y_1|$ and the the distance formula still holds since
$d(P_1, P_2) = |y_2 - y_1| = \sqrt{0 + (y_2 - y_1)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. □

Theorem 1.1.A (continued 2)

Proof (continued). Second, if the line joining P_1 and P_2 is horizontal, then the y-coordinate of P_1 equals the y-coordinate of P_2; that is, $y_1 = y_2$. In this case $d(P_1, P_2)$ is simply $|x_2 - x_1|$ and the the distance formula still holds since
$d(P_1, P_2) = |x_2 - x_1| = \sqrt{(x_2 - x_1)^2 + 0} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.





If the line joining P_1 and P_2 is vertical, then the x-coordinate of P_1 equals the x-coordinate of P_2; that is, $x_1 = x_2$. In this case $d(P_1, P_2)$ is simply $|y_2 - y_1|$ and the the distance formula still holds since
$d(P_1, P_2) = |y_2 - y_1| = \sqrt{0 + (y_2 - y_1)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. □

Page 7 Number 26

Page 7 Number 26. Find the distance between the points $P_1 = (2, -3)$ and $P_2 = (4, 2)$.

Solution. We use the distance formula

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where we have $P_1 = (x_1, y_1) = (2, -3)$ and $P_2 = (x_2, y_2) = (4, 2)$. So we have $x_1 = 2$, $y_1 = -3$, $x_2 = 4$, and $y_2 = 2$.

Page 7 Number 26

Page 7 Number 26. Find the distance between the points $P_1 = (2, -3)$ and $P_2 = (4, 2)$.

Solution. We use the distance formula

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where we have $P_1 = (x_1, y_1) = (2, -3)$ and $P_2 = (x_2, y_2) = (4, 2)$. So we have $x_1 = 2$, $y_1 = -3$, $x_2 = 4$, and $y_2 = 2$. Hence the distance is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(4-2)^2 + (2-(-3))^2}$$

$$= \sqrt{2^2 + (5)^2} = \sqrt{4 + 25} = \boxed{\sqrt{29}}.$$

□

Page 7 Number 26

Page 7 Number 26. Find the distance between the points $P_1 = (2, -3)$ and $P_2 = (4, 2)$.

Solution. We use the distance formula

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where we have $P_1 = (x_1, y_1) = (2, -3)$ and $P_2 = (x_2, y_2) = (4, 2)$. So we have $x_1 = 2$, $y_1 = -3$, $x_2 = 4$, and $y_2 = 2$. Hence the distance is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(4 - 2)^2 + (2 - (-3))^2}$$

$$= \sqrt{2^2 + (5)^2} = \sqrt{4 + 25} = \boxed{\sqrt{29}}.$$

□

Page 7 Number 34

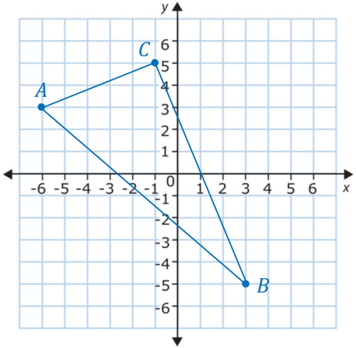
Page 7 Number 34. Plot each point $A = (-6, 3)$, $B = (3, -5)$, and $C = (-1, 5)$, and form the triangle ABC. Show that the triangle is a right triangle and give its area.

Solution. The points and triangle ABC are:

Page 7 Number 34

Page 7 Number 34. Plot each point $A = (-6, 3)$, $B = (3, -5)$, and $C = (-1, 5)$, and form the triangle ABC. Show that the triangle is a right triangle and give its area.

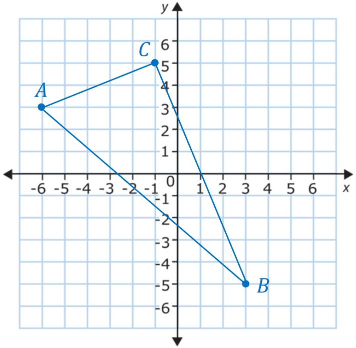
Solution. The points and triangle ABC are:



Page 7 Number 34

Page 7 Number 34. Plot each point $A = (-6, 3)$, $B = (3, -5)$, and $C = (-1, 5)$, and form the triangle ABC. Show that the triangle is a right triangle and give its area.

Solution. The points and triangle ABC are:



The lengths of the edges of ABC are given by the distances:

$$d(A, B) = \sqrt{(3 - (-6))^2 + (-5 - 3)^2} = \sqrt{81 + 64} = \sqrt{145},$$

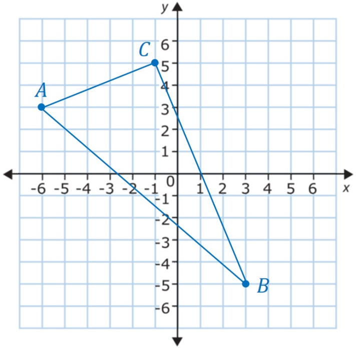
$$d(A, C) = \sqrt{(-1 - (-6))^2 + (5 - 3)^2} = \sqrt{25 + 4} = \sqrt{29}, \text{ and}$$

$$d(B, C) = \sqrt{(-1 - (3))^2 + (5 - (-5))^2} = \sqrt{16 + 100} = \sqrt{116}.$$

Page 7 Number 34

Page 7 Number 34. Plot each point $A = (-6, 3)$, $B = (3, -5)$, and $C = (-1, 5)$, and form the triangle ABC. Show that the triangle is a right triangle and give its area.

Solution. The points and triangle ABC are:



The lengths of the edges of ABC are given by the distances:

$$d(A, B) = \sqrt{(3 - (-6))^2 + (-5 - 3)^2}$$
$$= \sqrt{81 + 64} = \sqrt{145},$$
$$d(A, C) = \sqrt{(-1 - (-6))^2 + (5 - 3)^2}$$
$$= \sqrt{25 + 4} = \sqrt{29}, \text{ and}$$
$$d(B, C) = \sqrt{(-1 - (3))^2 + (5 - (-5))^2}$$
$$= \sqrt{16 + 100} = \sqrt{116}.$$

Page 7 Number 34 (continued)

Page 7 Number 34. Plot of the points each point $A = (-6, 3)$, $B = (3, -5)$, and $C = (-1, 5)$, and form the triangle ABC. Show that the triangle is a right triangle and give its area.

Solution (continued). By the Converse of the Pythagorean Theorem (see Appendix A.2. Geometry Essentials), ABC is a right triangle if the square of the length of one side is the sum of the squares of the lengths of the other two sides. We have $(\sqrt{145})^2 = (\sqrt{29})^2 + \sqrt{116}^2$ (since $145 = 29 + 116$) and so $d(A, B)^2 = d(A, C)^2 + d(B, C)^2$. Therefore, ABC is a right triangle with the right angle at vertex C and with side AB as the hypotenuse.

The area of a triangle is 1/2 the base times the height. With side AC as the base and side BC as the height, we then have the area $A = (1/2)d(A, C)d(B, C) = (1/2)\sqrt{29}\sqrt{116} = 58/2 = \boxed{29}$. □

Page 7 Number 34 (continued)

Page 7 Number 34. Plot of the points each point $A = (-6, 3)$, $B = (3, -5)$, and $C = (-1, 5)$, and form the triangle ABC. Show that the triangle is a right triangle and give its area.

Solution (continued). By the Converse of the Pythagorean Theorem (see Appendix A.2. Geometry Essentials), ABC is a right triangle if the square of the length of one side is the sum of the squares of the lengths of the other two sides. We have $(\sqrt{145})^2 = (\sqrt{29})^2 + \sqrt{116}^2$ (since $145 = 29 + 116$) and so $d(A, B)^2 = d(A, C)^2 + d(B, C)^2$. Therefore, ABC is a right triangle with the right angle at vertex C and with side AB as the hypotenuse.

The area of a triangle is 1/2 the base times the height. With side AC as the base and side BC as the height, we then have the area $A = (1/2)d(A, C)d(B, C) = (1/2)\sqrt{29}\sqrt{116} = 58/2 = \boxed{29}$. $\square$

Theorem 1.1.B

Theorem 1.1.B. The Midpoint Formula. The midpoint $M = (x, y)$ of the line segment from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$ is

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

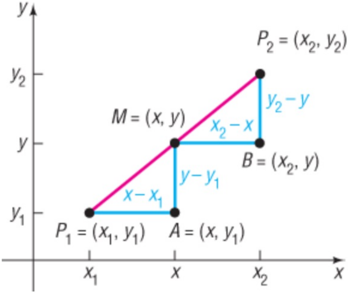
Proof. Consider the triangles P_1AM and MBP_2:

Theorem 1.1.B

Theorem 1.1.B. The Midpoint Formula. The midpoint $M = (x, y)$ of the line segment from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$ is

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

Proof. Consider the triangles P_1AM and MBP_2:



Theorem 1.1.B

Theorem 1.1.B. The Midpoint Formula. The midpoint $M = (x, y)$ of the line segment from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$ is

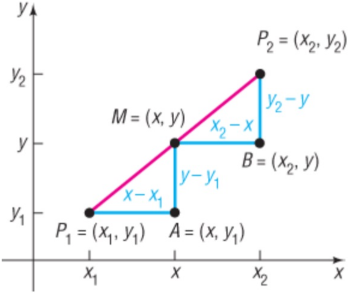
$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Proof. Consider the triangles P_1AM and MBP_2:

By classical Euclidean geometry, triangles P_1AM and MBP_2 are congruent (by angle-side-angle). So corresponding sides of the triangles are equal and

$$\begin{aligned} x - x_1 &= x_2 - x \\ 2x &= x_1 + x_2 \\ x &= \tfrac{x_1+x_2}{2} \end{aligned} \qquad \begin{aligned} y - y_1 &= y_2 - y \\ 2y &= y_1 + y_2 \\ y &= \tfrac{y_1+y_2}{2} \end{aligned}$$

□



Theorem 1.1.B

Theorem 1.1.B. The Midpoint Formula. The midpoint $M = (x, y)$ of the line segment from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$ is

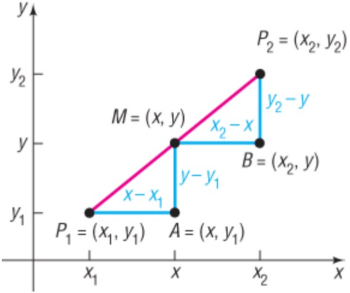
$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

Proof. Consider the triangles P_1AM and MBP_2:

By classical Euclidean geometry, triangles P_1AM and MBP_2 are congruent (by angle-side-angle). So corresponding sides of the triangles are equal and

$$\begin{aligned} x - x_1 &= x_2 - x \\ 2x &= x_1 + x_2 \\ x &= \tfrac{x_1+x_2}{2} \end{aligned} \qquad \begin{aligned} y - y_1 &= y_2 - y \\ 2y &= y_1 + y_2 \\ y &= \tfrac{y_1+y_2}{2} \end{aligned}$$

□



Page 7 Number 40

Page 7 Number 40. Find the midpoint of the line segment joining the points $P_1 = (2, -3)$ and $P_2 = (4, 2)$.

Solution. We use the midpoint formula

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

where $P_1 = (x_1, y_1) = (2, -3)$ and $P_2 = (x_2, y_2) = (4, 2)$. So we have $x_1 = 2$, $y_1 = -3$, $x_2 = 4$, and $y_2 = 2$.

Page 7 Number 40

Page 7 Number 40. Find the midpoint of the line segment joining the points $P_1 = (2, -3)$ and $P_2 = (4, 2)$.

Solution. We use the midpoint formula

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

where $P_1 = (x_1, y_1) = (2, -3)$ and $P_2 = (x_2, y_2) = (4, 2)$. So we have $x_1 = 2$, $y_1 = -3$, $x_2 = 4$, and $y_2 = 2$. So the midpoint is

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{2 + 4}{2}, \frac{-3 + 2}{2}\right)$$

$$= \left(\frac{6}{2}, \frac{-1}{2}\right) = \boxed{(3, -1/2)}.$$

Page 7 Number 40

Page 7 Number 40. Find the midpoint of the line segment joining the points $P_1 = (2, -3)$ and $P_2 = (4, 2)$.

Solution. We use the midpoint formula

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

where $P_1 = (x_1, y_1) = (2, -3)$ and $P_2 = (x_2, y_2) = (4, 2)$. So we have $x_1 = 2$, $y_1 = -3$, $x_2 = 4$, and $y_2 = 2$. So the midpoint is

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left(\frac{2 + 4}{2}, \frac{-3 + 2}{2} \right)$$

$$= \left(\frac{6}{2}, \frac{-1}{2} \right) = \boxed{(3, -1/2)}.$$

□

Page 7 Number 44

Page 7 Number 44. Find the midpoint of the line segment joining the points $P_1 = (a, a)$ and $P_2 = (0, 0)$.

Solution. We use the midpoint formula

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

where $P_1 = (x_1, y_1) = (a, a)$ and $P_2 = (x_2, y_2) = (0, 0)$. So we have $x_1 = a$, $y_1 = a$, $x_2 = 0$, and $y_2 = 0$.

Page 7 Number 44

Page 7 Number 44. Find the midpoint of the line segment joining the points $P_1 = (a, a)$ and $P_2 = (0, 0)$.

Solution. We use the midpoint formula

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

where $P_1 = (x_1, y_1) = (a, a)$ and $P_2 = (x_2, y_2) = (0, 0)$. So we have $x_1 = a$, $y_1 = a$, $x_2 = 0$, and $y_2 = 0$. So the midpoint is

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{a + 0}{2}, \frac{a + 0}{2}\right) = \boxed{\left(\frac{a}{2}, \frac{a}{2}\right)}.$$

□

Page 7 Number 44

Page 7 Number 44. Find the midpoint of the line segment joining the points $P_1 = (a, a)$ and $P_2 = (0, 0)$.

Solution. We use the midpoint formula

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

where $P_1 = (x_1, y_1) = (a, a)$ and $P_2 = (x_2, y_2) = (0, 0)$. So we have $x_1 = a$, $y_1 = a$, $x_2 = 0$, and $y_2 = 0$. So the midpoint is

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{a + 0}{2}, \frac{a + 0}{2}\right) = \boxed{\left(\frac{a}{2}, \frac{a}{2}\right)}.$$

□

Page 8 Number 56

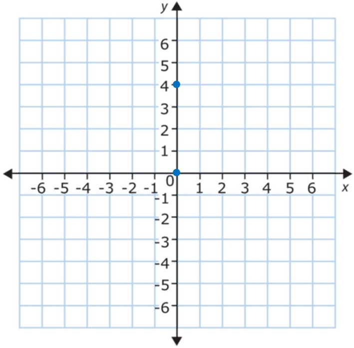
Page 8 Number 56. An *equilateral triangle* is one in which all three sides are of equal length. If two vertices of an equilateral triangle are $(0, 4)$ and $(0, 0)$, find the third vertex. How many of these triangles are possible?

Solution. Consider the graph of the two points:

Page 8 Number 56

Page 8 Number 56. An *equilateral triangle* is one in which all three sides are of equal length. If two vertices of an equilateral triangle are $(0, 4)$ and $(0, 0)$, find the third vertex. How many of these triangles are possible?

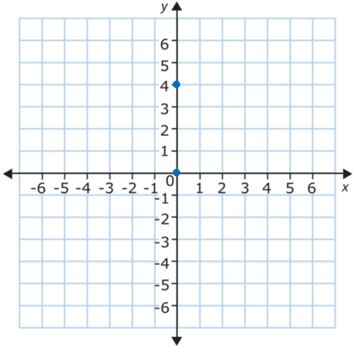
Solution. Consider the graph of the two points:



Page 8 Number 56

Page 8 Number 56. An *equilateral triangle* is one in which all three sides are of equal length. If two vertices of an equilateral triangle are $(0, 4)$ and $(0, 0)$, find the third vertex. How many of these triangles are possible?

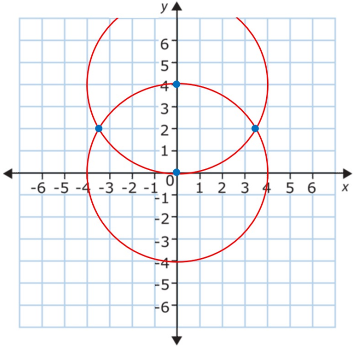
Solution. Consider the graph of the two points:



Page 8 Number 56

Page 8 Number 56. An *equilateral triangle* is one in which all three sides are of equal length. If two vertices of an equilateral triangle are $(0, 4)$ and $(0, 0)$, find the third vertex. How many of these triangles are possible?

Solution. Consider the graph of the two points:



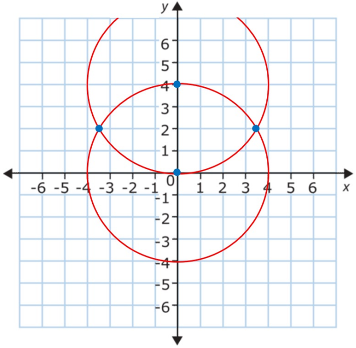
For an equilateral triangle, we need a third point that is also a distance of 4 from both given points. So we draw circles of radius 4 centered at each point.

We see that there are two possible choices for the third point.
We now find the coordinates of these two points.

Page 8 Number 56

Page 8 Number 56. An *equilateral triangle* is one in which all three sides are of equal length. If two vertices of an equilateral triangle are $(0, 4)$ and $(0, 0)$, find the third vertex. How many of these triangles are possible?

Solution. Consider the graph of the two points:



For an equilateral triangle, we need a third point that is also a distance of 4 from both given points. So we draw circles of radius 4 centered at each point.

We see that there are two possible choices for the third point.
We now find the coordinates of these two points.

Page 8 Number 56 (continued)

Page 8 Number 56. An *equilateral triangle* is one in which all three sides are of equal length. If two vertices of an equilateral triangle are $(0,4)$ and $(0,0)$, find the third vertex. How many of these triangles are possible?

Solution (continued). Let (x,y) be a point distance 4 from both $(0,4)$ and $(0,0)$. Then from the distance formula, $4 = \sqrt{(x-0)^2+(y-4)^2} = \sqrt{(x-0)^2+(y-0)^2}$ or, squaring,

$$16 = x^2+(y-4)^2 = x^2+y^2. \qquad (*)$$

So we need $x^2+y^2-8y+16 = x^2+y^2$ or $-8y+16=0$ or $y=2$. From $(*)$ with $y=2$, we need (say) $16 = x^2+(2)^2$ or $x^2=12$ or $\sqrt{x^2}=\sqrt{12}$ or $|x| = \sqrt{12} = 2\sqrt{3}$. So we must have $x = \pm 2\sqrt{3}$. The two points which make an equilateral triangle are

$$\boxed{(x,y) = (2\sqrt{3},2) \text{ and } (x,y) = (-2\sqrt{3},2)}. \quad \square$$

Page 8 Number 56 (continued)

Page 8 Number 56. An *equilateral triangle* is one in which all three sides are of equal length. If two vertices of an equilateral triangle are $(0, 4)$ and $(0, 0)$, find the third vertex. How many of these triangles are possible?

Solution (continued). Let (x, y) be a point distance 4 from both $(0, 4)$ and $(0, 0)$. Then from the distance formula,
$4 = \sqrt{(x-0)^2 + (y-4)^2} = \sqrt{(x-0)^2 + (y-0)^2}$ or, squaring,

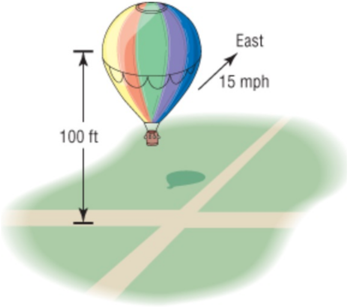
$$16 = x^2 + (y-4)^2 = x^2 + y^2. \qquad (*)$$

So we need $x^2 + y^2 - 8y + 16 = x^2 + y^2$ or $-8y + 16 = 0$ or $y = 2$. From $(*)$ with $y = 2$, we need (say) $16 = x^2 + (2)^2$ or $x^2 = 12$ or $\sqrt{x^2} = \sqrt{12}$ or $|x| = \sqrt{12} = 2\sqrt{3}$. So we must have $x = \pm 2\sqrt{3}$. The two points which make an equilateral triangle are

$\boxed{(x, y) = (2\sqrt{3}, 2) \text{ and } (x, y) = (-2\sqrt{3}, 2)}$. □

Page 8 Number 68

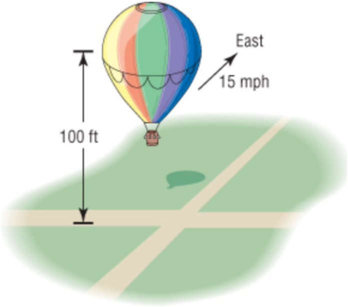
Page 8 Number 68. A hot-air balloon, headed due east at an average speed of 15 miles per hour and at a constant altitude of 100 feet, passes over an intersection (see the figure). Find an expression for the distance d (measured in feet) from the balloon to the intersection t seconds later. HINT: 1 mile is 5280 feet and 1 hour is 3600 seconds.



Solution. We need to use the distance formula: (distance) = (rate)(time).

Page 8 Number 68

Page 8 Number 68. A hot-air balloon, headed due east at an average speed of 15 miles per hour and at a constant altitude of 100 feet, passes over an intersection (see the figure). Find an expression for the distance d (measured in feet) from the balloon to the intersection t seconds later. HINT: 1 mile is 5280 feet and 1 hour is 3600 seconds.



Solution. We need to use the distance formula: (distance) = (rate)(time).

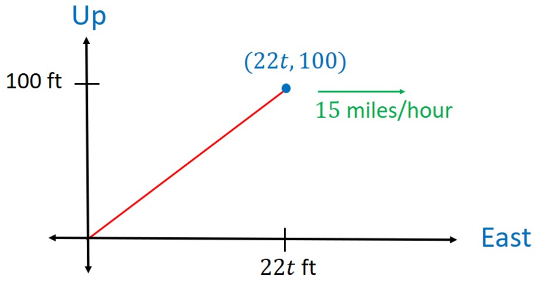
Page 8 Number 68 (continued 1)

Solution (continued). The horizontal distance traveled after t seconds is

$$x = (15 \text{ miles/hour})\left(\frac{5280 \text{ feet}}{1 \text{ mile}}\right)(t \text{ seconds})\left(\frac{1 \text{ hour}}{3600 \text{ seconds}}\right)$$

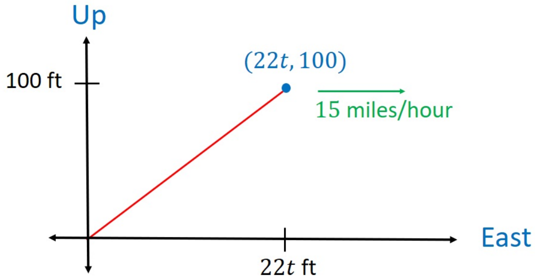
$$= \frac{(15)(5280)t}{3600} \text{ feet} = 22t \text{ feet}.$$

So at time t seconds we have:



Page 8 Number 68 (continued 2)

Solution (continued).



So with the origin of the above coordinate system at the intersection, the distance from the intersection at $(0, 0)$ to the balloon at $(22t, 100)$ (with all distance units in feet) is

$$\sqrt{(22t-0)^2+(100-0)^2} = \boxed{\sqrt{484t^2+10000} = d}.$$

□