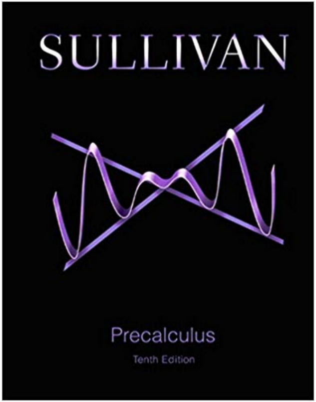
Precalculus 1 (Algebra)

Chapter 1. Graphs

1.4. Circles—Exercises, Examples, Proofs



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Solution. The standard form of an equation of a circle with radius r and center (h, k) is $(x - h)^2 + (y - k)^2 = r^2$. So we have $(x-2)^2 + (y-(-3))^2 = (4)^2$ or $\boxed{(x-2)^2 + (y+3)^2 = 16}$. The graph is:

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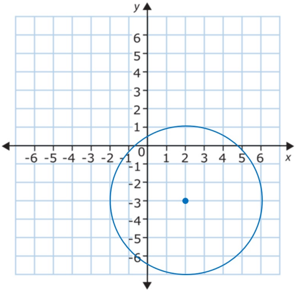

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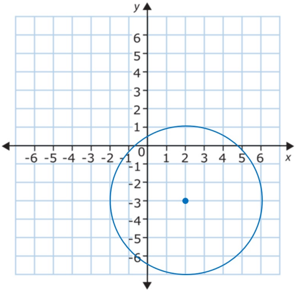

Page 38 Number 24

Page 38 Number 24. Consider the standard form for the equation of a circle: $x^2 + (y-1)^2 = 1$. Find the center, radius, intercepts (if any), and graph the circle.

Solution. The standard form of an equation of a circle with radius r and center (h, k) is $(x-h)^2 + (y-k)^2 = r^2$. So circle $x^2 + (y-1)^2 = 1$ has $\boxed{\text{center } (h,k) = (0,1) \text{ and radius } r = \sqrt{1} = 1}$.

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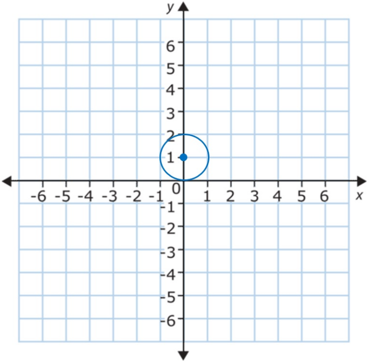
Solution (continued). The graph is:



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Page 38 Number 36

Page 38 Number 36. Consider the equation of a circle: $3x^2 + 3y^2 - 12y = 0$. Find the center, radius, intercepts (if any), and graph the circle.

Solution. We have $3x^2 + 3y^2 - 12y = 0$ or, dividing by 3, $x^2 + y^2 - 4y = 0$. Completing the square on y (see Appendix A.3. Polynomials) we have $x^2 + y^2 - 4y + (-4/2)^2 = 0 + (-4/2)^2$ or $x^2 + y^2 - 4y + 4 = 4$ or $x^2 + (y-2)^2 = 2^2$. Since the standard form of an equation of a circle with radius r and center (h, k) is $(x-h)^2 + (y-k)^2 = r^2$ then we see that the $\boxed{\text{center is } (h, k) = (0, 2) \text{ and the radius is } r = 2}$.

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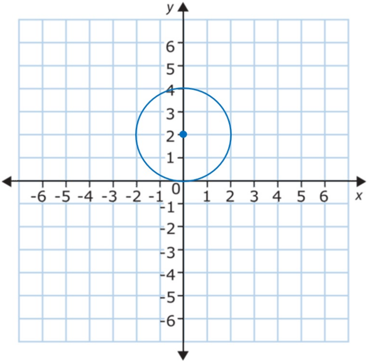
Page 38 Number 36 (continued)

Solution (continued).



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Solution. Notice that the graph of the circle must be:

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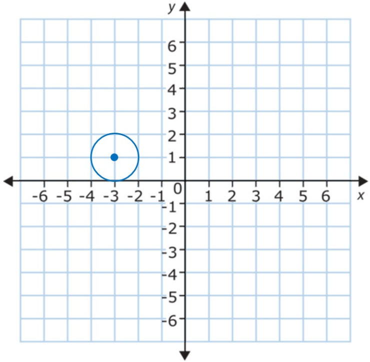

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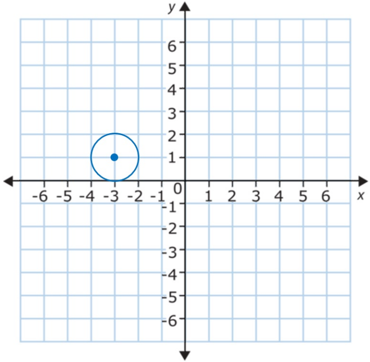

So the radius must be $r = 1$, and since the center is $(h, k) = (-3, 1)$ then standard form of an equation of the circle is $(x - h)^2 + (y - k)^2 = r^2$ or $(x - (-3))^2 + (y - (1))^2 = 1^2$ or $\boxed{(x + 3)^2 + (y - 1)^2 = 1}$. □

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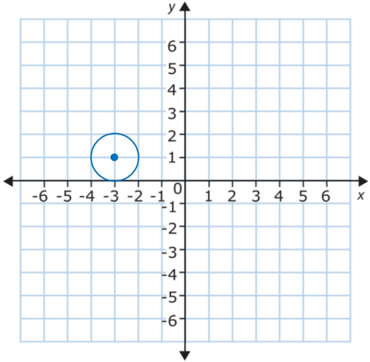


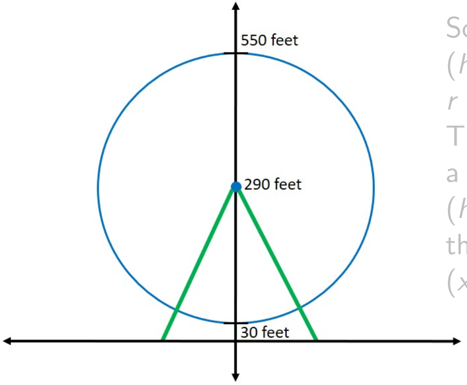
So the radius must be $r = 1$, and since the center is $(h, k) = (-3, 1)$ then standard form of an equation of the circle is $(x - h)^2 + (y - k)^2 = r^2$ or $(x - (-3))^2 + (y - (1))^2 = 1^2$ or $\boxed{(x + 3)^2 + (y - 1)^2 = 1}$. □

Page 39 Number 52

Page 39 Number 52. Opening in 2014 in Las Vegas, The High Roller observation wheel has a maximum height of 550 feet and a diameter of 520 feet, with one full rotation taking approximately 30 minutes. Find an equation for the wheel if the center of the wheel is on the y-axis and the x-axis is at ground level.

Page 39 Number 52 (page)

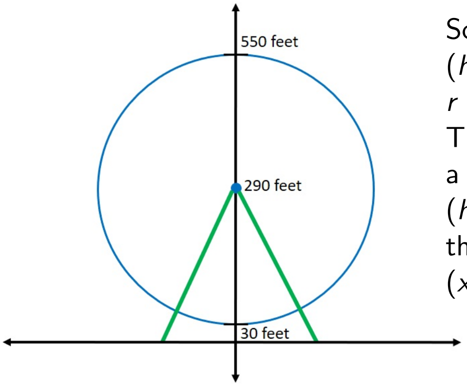
Solution. The maximum height of the wheel is 550 feet and the diameter is 520, so the minimum height of the wheel is
550 feet $-$ 520 feet $=$ 30 feet. So the average height of the wheel is $(550+30)/2 = 290$ feet, and this locates the center of the wheel. Schematically we have:



So the center of the wheel is $(h,k) = (0,290)$ and the radius is $r = 290 - 30 = 550 - 290 = 260$. The standard form of an equation of a circle with radius r and center (h,k) is $(x-h)^2 + (y-k)^2 = r^2$ so the wheel has equation $(x-0)^2 + (y-290)^2 = 260^2$ or $\boxed{x^2 + (y-290)^2 = 67,600}$.

Page 39 Number 52 (page)

Solution. The maximum height of the wheel is 550 feet and the diameter is 520, so the minimum height of the wheel is 550 feet − 520 feet = 30 feet. So the average height of the wheel is $(550 + 30)/2 = 290$ feet, and this locates the center of the wheel. Schematically we have:



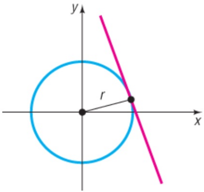
So the center of the wheel is $(h, k) = (0, 290)$ and the radius is $r = 290 - 30 = 550 - 290 = 260$. The standard form of an equation of a circle with radius r and center (h, k) is $(x - h)^2 + (y - k)^2 = r^2$ so the wheel has equation $(x - 0)^2 + (y - 290)^2 = 260^2$ or

$$\boxed{x^2 + (y - 290)^2 = 67,600}.$$

□

Page 39 Number 54

Page 39 Number 54. The *tangent line* to a circle may be defined as the line that intersects the circle in a single point, called the *point of tangency*:



If the equation of the circle is $x^2 + y^2 = r^2$ and the equation of the tangent line is $y = mx + b$, show that

(a) $r^2(1 + m^2) = b^2$. HINT: You need the quadratic formula here.

(b) The point of tangency is $\left(\dfrac{-r^2 m}{b}, \dfrac{r^2}{b}\right)$.

(c) The tangent line is perpendicular to the line containing the center of the circle and the point of tangency.

Page 39 Number 54 (continued 1)

(a) $r^2(1+m^2) = b^2$. HINT: You need the quadratic formula here.

Solution. (a) At the point of tangency (x, y) both relationships $x^2 + y^2 = r^2$ and $y = mx + b$ hold. Substituting the second equation into the first we get $x^2 + (mx+b)^2 = r^2$ or $x^2 + m^2x^2 + 2mxb + b^2 - r^2 = 0$ or $(1+m^2)x^2 + (2mb)x + (b^2 - r^2) = 0$. Solving for x using the quadratic formula gives $x = \dfrac{-(2mb) \pm \sqrt{(2mb)^2 - 4(1+m^2)(b^2-r^2)}}{2(1+m^2)}$.

Page 39 Number 54 (continued 1)

(a) $r^2(1+m^2) = b^2$. HINT: You need the quadratic formula here.

Solution. **(a)** At the point of tangency (x, y) both relationships $x^2 + y^2 = r^2$ and $y = mx + b$ hold. Substituting the second equation into the first we get $x^2 + (mx+b)^2 = r^2$ or $x^2 + m^2x^2 + 2mxb + b^2 - r^2 = 0$ or $(1+m^2)x^2 + (2mb)x + (b^2 - r^2) = 0$. Solving for x using the quadratic formula gives $x = \dfrac{-(2mb) \pm \sqrt{(2mb)^2 - 4(1+m^2)(b^2-r^2)}}{2(1+m^2)}$. Since we see from the picture that there is exactly one x-value satisfying this, the term under the square root (the "discriminant") must be 0 (see Appendix A.6. Solving Equations, Note A.6.C); that is, $(2mb)^2 - 4(1+m^2)(b^2-r^2) = 0$. From this equation, we have $4m^2b^2 - 4(b^2 - r^2 + m^2b^2 - m^2r^2) = 0$ or $-4b^2 + 4r^2 + 4m^2r^2 = 0$ or $-b^2 + r^2 + m^2r^2 = 0$ or $r^2(1+m^2) = b^2$, as claimed. □

Page 39 Number 54 (continued 1)

(a) $r^2(1+m^2) = b^2$. HINT: You need the quadratic formula here.

Solution. (a) At the point of tangency (x, y) both relationships $x^2 + y^2 = r^2$ and $y = mx + b$ hold. Substituting the second equation into the first we get $x^2 + (mx+b)^2 = r^2$ or $x^2 + m^2x^2 + 2mxb + b^2 - r^2 = 0$ or $(1+m^2)x^2 + (2mb)x + (b^2 - r^2) = 0$. Solving for x using the quadratic formula gives $x = \dfrac{-(2mb) \pm \sqrt{(2mb)^2 - 4(1+m^2)(b^2-r^2)}}{2(1+m^2)}$. Since we see from the picture that there is exactly one x-value satisfying this, the term under the square root (the "discriminant") must be 0 (see Appendix A.6. Solving Equations, Note A.6.C); that is, $(2mb)^2 - 4(1+m^2)(b^2-r^2) = 0$. From this equation, we have $4m^2b^2 - 4(b^2 - r^2 + m^2b^2 - m^2r^2) = 0$ or $-4b^2 + 4r^2 + 4m^2r^2 = 0$ or $-b^2 + r^2 + m^2r^2 = 0$ or $r^2(1+m^2) = b^2$, as claimed. □

Page 39 Number 54 (continued 2)

(b) The point of tangency is $\left(\frac{-r^2 m}{b}, \frac{r^2}{b}\right)$.

Solution. (b) From part (a)
$x = \frac{-(2mb) \pm \sqrt{(2mb)^2 - 4(1+m^2)(b^2-r^2)}}{2(1+m^2)}$ where
$(2mb)^2 - 4(1+m^2)(b^2-r^2) = 0$, so $x = \frac{-(2mb)}{2(1+m^2)} = \frac{-mb}{1+m^2}$. By the conclusion of part (a), $r^2(1+m^2) = b^2$ or $1+m^2 = b^2/r^2$ so we then have $x = \frac{-mb}{1+m^2} = \frac{-mb}{b^2/r^2} = \frac{-r^2 m}{b}$.

Since $y = mx + b$, we can find y corresponding to this x-value as
$y = m(-r^2 m/b) + b = -r^2 m^2/b + b^2/b = (-r^2 m^2 + b^2)/b$. By part (a)
$r^2(1+m^2) = b^2$, so
$y = (-r^2 m^2 + b^2)/b = (-r^2 m^2 + r^2(1+m^2))/b = r^2/b$. So
the point of tangency is $(x, y) = (-r^2 m/b, r^2/b)$, as claimed. □

Page 39 Number 54 (continued 2)

(b) The point of tangency is $\left(\frac{-r^2 m}{b}, \frac{r^2}{b}\right)$.

Solution. **(b)** From part (a)

$x = \frac{-(2mb) \pm \sqrt{(2mb)^2 - 4(1+m^2)(b^2-r^2)}}{2(1+m^2)}$ where

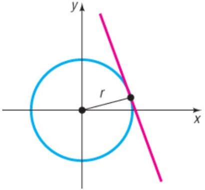
$(2mb)^2 - 4(1+m^2)(b^2-r^2) = 0$, so $x = \frac{-(2mb)}{2(1+m^2)} = \frac{-mb}{1+m^2}$. By the conclusion of part (a), $r^2(1+m^2) = b^2$ or $1+m^2 = b^2/r^2$ so we then have $x = \frac{-mb}{1+m^2} = \frac{-mb}{b^2/r^2} = \frac{-r^2 m}{b}$.

Since $y = mx + b$, we can find y corresponding to this x-value as $y = m(-r^2 m/b) + b = -r^2 m^2/b + b^2/b = (-r^2 m^2 + b^2)/b$. By part (a) $r^2(1+m^2) = b^2$, so
$y = (-r^2 m^2 + b^2)/b = (-r^2 m^2 + r^2(1+m^2))/b = r^2/b$. So

$\boxed{\text{the point of tangency is } (x,y) = (-r^2 m/b, r^2/b)}$, as claimed. □

Page 39 Number 54 (continued 3)

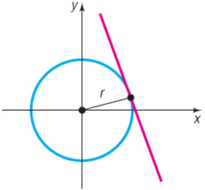
(c) The tangent line is perpendicular to the line containing the center of the circle and the point of tangency.



Solution. (c) Since the tangent line is $y = mx + b$, then its slope is m. Since the line containing the center of the circle and the point of tangency passes through points $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (-r^2m/b, r^2/b)$, then its slope is $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{r^2/b - 0}{-r^2m/b - 0} = \dfrac{r^2/b}{-r^2m/b} = \dfrac{1}{-m}$. Since $(m)\left(\dfrac{1}{-m}\right) = -1$, then by Theorem 1.3.F, "Criterion for Perpendicular Lines," the tangent line and the line containing the center of the circle and the point of tangency are perpendicular, as claimed. □

Page 39 Number 54 (continued 3)

(c) The tangent line is perpendicular to the line containing the center of the circle and the point of tangency.



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