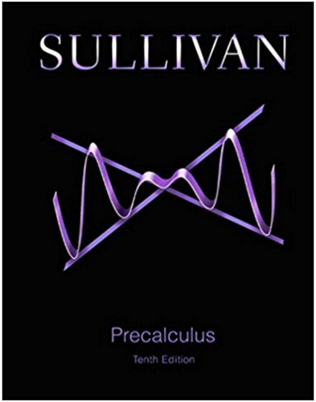
Precalculus 1 (Algebra)

Chapter 2. Functions and Their Graphs

2.3. Properties of Functions—Exercises, Examples, Proofs



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Page 78 Number 26(a),(b),(d)

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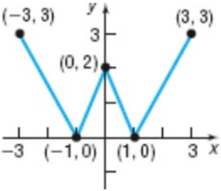

Solution. (a) Since the points $(-1, 0)$, $(0, 2)$, and $(1, 0)$ are the only points on the graph of f which are on the axes, then the $\boxed{x\text{-intercepts are } (-1, 0) \text{ and } (1, 0)}$, and the $\boxed{y\text{-intercept is } (0, 2)}$.

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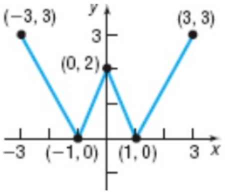

Solution. **(a)** Since the points $(-1, 0)$, $(0, 2)$, and $(1, 0)$ are the only points on the graph of f which are on the axes, then the $\boxed{x\text{-intercepts are } (-1, 0) \text{ and } (1, 0)}$, and the $\boxed{y\text{-intercept is } (0, 2)}$.

(b) The domain is the set of x values where f is defined, so the $\boxed{\text{domain of } f \text{ is } [-3, 3]}$. The range is the set of $y = f(x)$ values, so the $\boxed{\text{range of } f \text{ is } [0, 3]}$.

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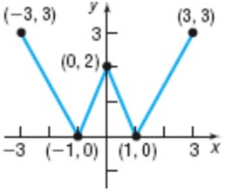

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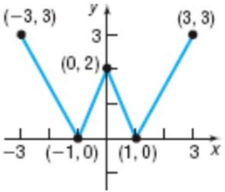

Solution (continued). (d) Since f is symmetric with respect to the y-axis, then by Theorem 2.3.A $\boxed{f \text{ is an even function}}$. Since f is not symmetric with respect to the origin (for example, there are not points in Quadrant III corresponding to the points in Quadrant I), then by Theorem 2.3.A $\boxed{f \text{ is not an odd function}}$. □

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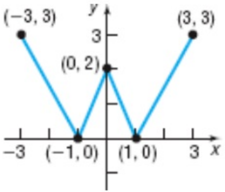

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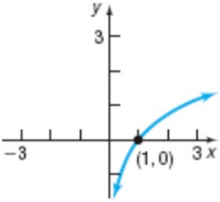

Solution. **(a)** Since the point $(1, 0)$ is the only point on the graph of f which is on an axis, then the $\boxed{x\text{-intercept is } (1, 0)}$, and $\boxed{\text{there are no } y\text{-intercepts}}$. □

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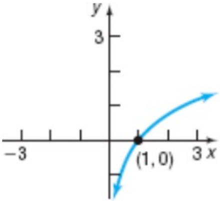

Solution. **(a)** Since the point $(1, 0)$ is the only point on the graph of f which is on an axis, then the $\boxed{x\text{-intercept is } (1, 0)}$, and $\boxed{\text{there are no } y\text{-intercepts}}$.

(b) The domain is the set of x values where f is defined, so (assuming the graph is extended according to the pattern we see) the $\boxed{\text{domain of } f \text{ is } (0, \infty)}$.

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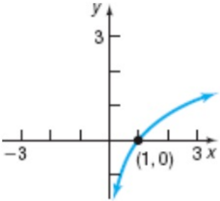

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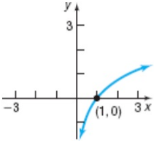

Solution (continued). **(b)** The range is the set of $y = f(x)$ values, so (assuming the graph is extended according to the pattern we see) the $\boxed{\text{range of } f \text{ is } (-\infty, \infty)}$ (here we assume that the graph continues to go upward without bound, though it is possible that it could have a horizontal asymptote and that there is an upper bound on the range). □

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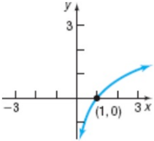

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(d) Since f is not symmetric with respect to the y-axis (for example, there are not points in Quadrant II corresponding to the points in Quadrant I), then by Theorem 2.3.A $\boxed{f \text{ is not an even function}}$. Since f is not symmetric with respect to the origin (for example, there are not points in Quadrant III corresponding to the points in Quadrant I), then by Theorem 2.3.A $\boxed{f \text{ is not an odd function}}$. □

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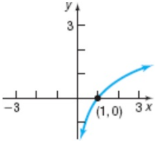

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Page 78 Number 26(c)

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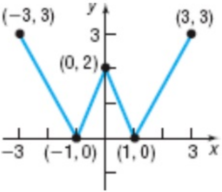

Solution. Since a function is increasing when it is going "uphill" (as read from left to right) then $\boxed{f \text{ is increasing on the open intervals } (-1, 0) \text{ and } (1, 3)}$. Since a function is decreasing when it is going "downhill" (as read from left to right) then $\boxed{f \text{ is decreasing on the open intervals } (-3, -1) \text{ and } (0, 1)}$. □

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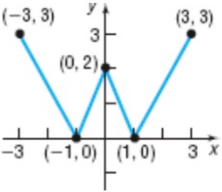

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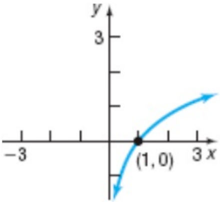

Solution. Since a function is increasing when it is going "uphill" (as read from left to right) then $\boxed{f \text{ is increasing on the open interval } (0, \infty)}$ (that is, f is increasing on its domain). □

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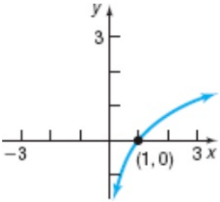

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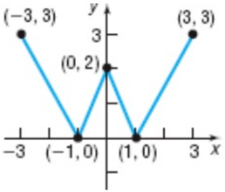

Solution. (a) The idea that f has a local minimum at an x value means that f is lesser at x than it is at "nearby" x values. So from the graph of f we see that it has a $\boxed{\text{local minimum of 0 at } x = -1 \text{ and at } x = 1}$. □

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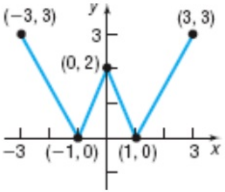

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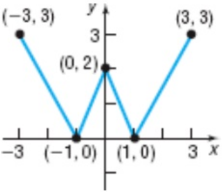

Solution (continued). (b) The idea that f has a local maximum at an x value means that f is greater at x than it is at "nearby" x values. So from the graph of f we see that it has $\boxed{\text{local maxima of 2 and 3}}$.

$\boxed{\text{The maximum 2 occurs at } x = 0}$.

$\boxed{\text{The maximum 3 occurs at } x = -3 \text{ and at } x = 3}$.

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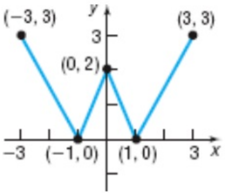

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Page 79 Number 50

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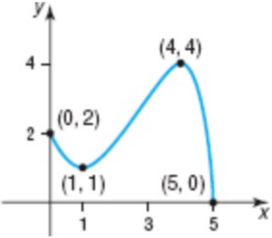

Solution. The idea that f has an absolute maximum at an x value means that the highest point on the graph of f occurs at the x value. So from the graph of f we see that it has an $\boxed{\text{absolute maximum of 4 at } x = 4}$. It also has a $\boxed{\text{local maximum of 2 at } x = 0}$.

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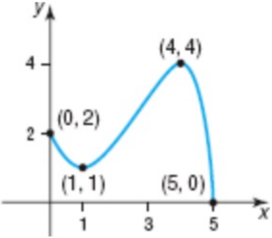

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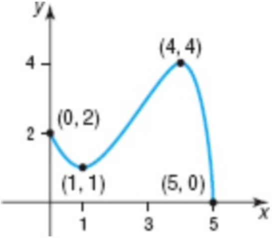

Solution (continued). The idea that f has an absolute minimum at an x value means that the lowest point on the graph of f occurs at the x value. So from the graph of f we see that it has an $\boxed{\text{absolute minimum of 0 at } x = 5}$. It also has a $\boxed{\text{local minimum of 1 at } x = 1}$. □

Page 79 Number 50 (continued)

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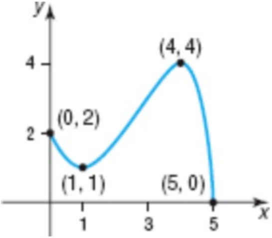

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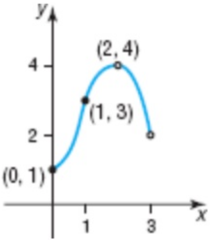

Solution. The idea that f has an absolute minimum at an x value means that the lowest point on the graph of f occurs at the x value. So from the graph of f we see that it has an $\boxed{\text{absolute minimum of 1 at } x = 0}$.

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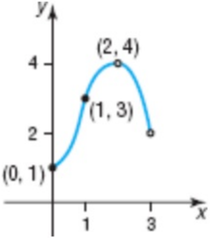
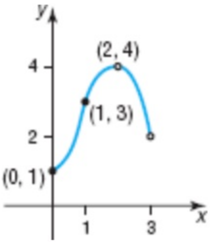


Solution. The idea that f has an absolute minimum at an x value means that the lowest point on the graph of f occurs at the x value. So from the graph of f we see that it has an $\boxed{\text{absolute minimum of 1 at } x = 0}$.

Page 79 Number 52 (continued)



Solution (continued). This function is more complicated than the ones we have dealt with above. One might think that it has an absolute maximum of 4 at $x = 2$, but this is not the case since f is not defined at $x = 2$. In fact, $\boxed{f \text{ has no absolute maximum}}$. Similarly, the point $(3, 2)$ does not correspond to a local minimum since this point is not on the graph of f. □

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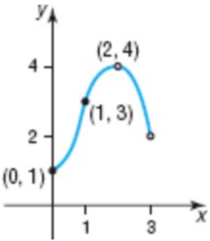

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Page 80 Number 72

Page 80 Number 72. Consider $g(x) = x^2 + 1$. (a) Find the average rate of change from -1 to 2. (b) Find an equation of the secant line containing $(-1, g(-1))$ and $(2, g(2))$.

Solution. (a) By definition, the average rate of change from a to b is

$$\frac{\Delta y}{\Delta x} = \frac{g(b) - g(a)}{b - a} = \frac{g(2) - g(-1)}{(2) - (-1)}$$

$$= \frac{((2)^2 + 1) - ((-1)^2 + 1)}{(2) - (-1)} = \frac{5 - 2}{3} = \boxed{1}.$$

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(b) By Theorem 2.3.C, "Slope of a Secant Line," the secant line has slope equal to the average rate of change so $m_{\text{sec}} = 1$. By the point-slope formula for a line, the secant line is $y - y_1 = m(x - x_1)$ or $y - g(-1) = 1(x - (-1))$ or $y - ((-1)^2 + 1) = x + 1$ or $y - 2 = x + 1$ or $\boxed{y = x + 3}$. □

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Page 80 Number 82

Page 80 Number 82. The size of the total debt owed by the United States federal government continues to grow. In fact, according to the Department of the Treasury, the debt per person living in the United States is approximately $53,000 (or over $140,000 per U.S. household). The following data represent the U.S. debt (in billions of dollars) for the years 2001-2013. Since the debt D depends on the year y, and each input corresponds to exactly one output, the debt is a function of the year. So $D(y)$ represents the debt for each year y.

Year	Debt	Year	Debt
2001	5807	2008	10,025
2002	6228	2009	11,910
2003	6783	2010	13,562
2004	7379	2011	14,790
2005	7933	2012	16,066
2006	8507	2013	16,738
2007	9008		

Page 80 Number 82 (continued 1)

(a) Plot the points $(2001, 5807)$, $(2002, 6228)$, and so on in a Cartesian plane.

(b) Draw a line segment from the point $(2001, 5807)$ to $(2006, 8507)$. What does the slope of this line segment represent?

(c) Find the average rate of change of the debt from 2002 to 2004.

(d) Find the average rate of change of the debt from 2006 to 2008.

(e) Find the average rate of change of the debt from 2010 to 2012.

(f) What appears to be happening to the average rate of change as time passes?

Page 80 Number 82 (continued 2)

(a) Plot the points $(2001, 5807)$, $(2002, 6228)$, and so on in a Cartesian plane.

(b) Draw a line segment from the point $(2001, 5807)$ to $(2006, 8507)$. What does the slope of this line segment represent?

Solution. (a), **(b)**

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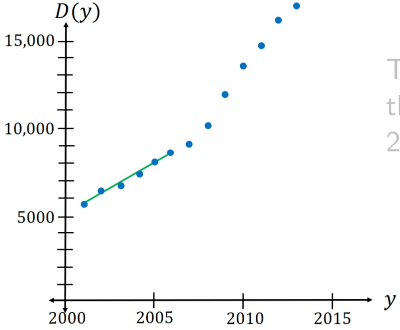

The slope of the line segment represents the average rate of change of debt from 2001 to 2006.

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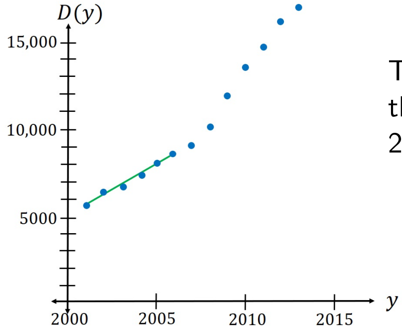

The slope of the line segment represents the average rate of change of debt from 2001 to 2006.

Page 80 Number 82 (continued 3)

(c) Find the average rate of change of the debt from 2002 to 2004.

(d) Find the average rate of change of the debt from 2006 to 2008.

Solution. **(c)** For the average rate of change of the debt from 2002 to 2004, we use the points $(2002, 6228)$ and $(2004, 7379)$. The average rate of change is

$$\frac{\Delta D}{\Delta y} = \frac{7379 - 6228}{2004 - 2002} = \frac{1151}{2} = 575.5 \text{ billions of dollars/year.} \quad \square$$

Page 80 Number 82 (continued 3)

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$$\frac{\Delta D}{\Delta y} = \frac{10{,}025 - 8507}{2008 - 2006} = \frac{1518}{2} = 759 \text{ billions of dollars/year.} \quad \square$$

Page 80 Number 82 (continued 3)

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Page 80 Number 82 (continued 4)

(e) Find the average rate of change of the debt from 2010 to 2012.

(f) What appears to be happening to the average rate of change as time passes?

Solution. (e) For the average rate of change of the debt from 2010 to 2012, we use the points (2010, 13,562) and (2012, 16,066). The average rate of change is

$$\frac{\Delta D}{\Delta y} = \frac{16{,}066 - 13{,}562}{2012 - 2010} = \frac{2504}{2} = 1252 \text{ billions of dollars/year.} \qquad \square$$

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(f) The average rate of change is increasing as time passes. The average rates of change calculated above are based on evenly spaced times, so we see that the rate of change itself has an increasing rate of change. You will explore these ideas in Calculus 1 with derivatives. $\square$

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