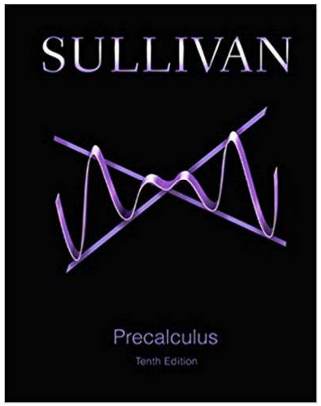
Precalculus 1 (Algebra)

Chapter 2. Functions and Their Graphs

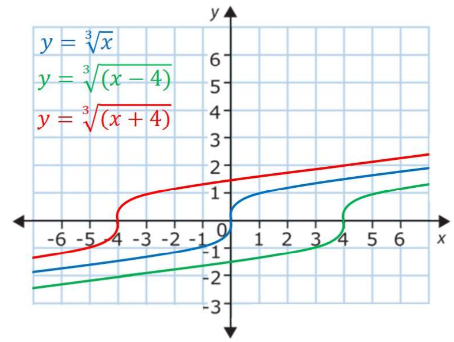
2.5. Graphing Techniques: Transformations—Exercises, Examples, Proofs



Example. Shifts

Example. Find a formula for the function whose graph is the graph of $y = \sqrt[3]{x}$ but is **(a)** shifted to the right 4 units, **(b)** shifted to the left 4 units, **(c)** shifted up 4 units, **(d)** shifted down 4 units.

Solution. (a) To shift to the right 4 units we replace x by $x - 4$ to get $y = \sqrt[3]{x-4}$. **(b)** To shift to the left 4 units we replace x by $x - (-4) = x + 4$ to get $y = \sqrt[3]{x+4}$.



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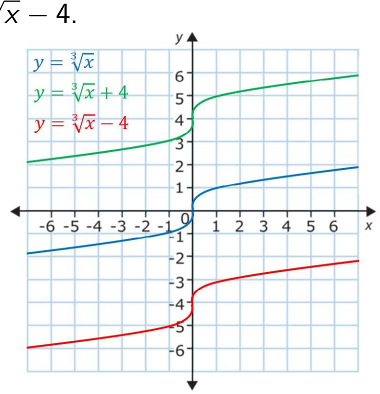
Example. Shifts

Example. Find a formula for the function whose graph is the graph of $y = \sqrt[3]{x}$ but is **(a)** shifted to the right 4 units, **(b)** shifted to the left 4 units, **(c)** shifted up 4 units, **(d)** shifted down 4 units.

Solution. (c) To shift up 4 units we add 4 to the original function to get $y = \sqrt[3]{x} + 4$. **(d)** To shift down 4 units we subtract 4 to the original function to get $y = \sqrt[3]{x} - 4$.



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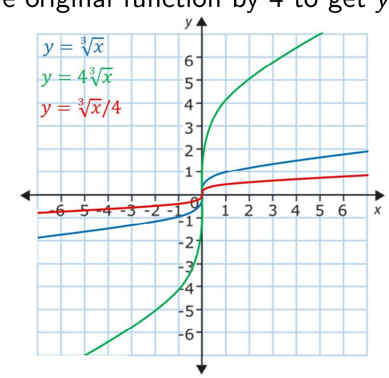
Example. Vertical Stretches/Compression

Example. Find a formula for the function whose graph is the graph of $y = \sqrt[3]{x}$ but is **(a)** vertically stretched where $a = 4$, **(b)** vertically compressed where $a = 1/4$.

Solution. (a) To vertically stretch with $a = 4$, we multiply the original function function by 4 to get $y = 4\sqrt[3]{x}$. **(b)** To vertically compress with $a = 1/4$, we divide the original function by 4 to get $y = \sqrt[3]{x}/4$.



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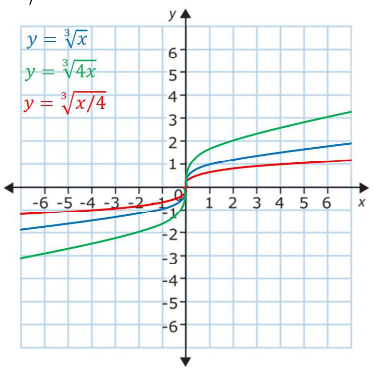
Example. Horizontal Stretches/Compression

Example. Find a formula for the function whose graph is the graph of $y = \sqrt[3]{x}$ but is **(a)** horizontally compressed where $a = 4$, **(b)** horizontally stretched where $a = 1/4$.

Solution. (a) To horizontally compress with $a = 4$, we replace x with $4x$ to get $y = \sqrt[3]{4x}$. **(b)** To horizontally stretch with $a = 1/4$, we replace x with $x/4$ to get $y = \sqrt[3]{x/4}$.



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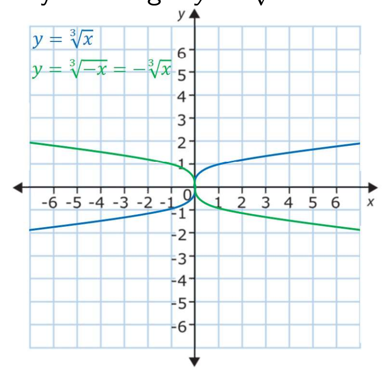
Example. Reflections

Example. Find a formula for the function whose graph is the graph of $y = \sqrt[3]{x}$ but is **(a)** reflected about the x-axis, **(b)** reflected about the y-axis.

Solution. (a) To reflect a function about the x-axis we multiply to original function by -1 to get $y = -\sqrt[3]{x}$. **(b)** To reflect a function about the y-axis we replace x by $-x$ to get $y = \sqrt[3]{-x} = -\sqrt[3]{x}$.



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Page 103 Number 32

Page 103 Number 32. If $(3, 6)$ is a point on the graph of $y = f(x)$, which of the following points must be on the graph of $y = f(-x)$? **(a)** $(6, 3)$, **(b)** $(6, -3)$, **(c)** $(3, -6)$, **(d)** $(-3, 6)$.

Solution. Since $(3, 6)$ is on the graph of f, then $6 = f(3)$ and so $6 = f(-(-3))$ and the point $(-3, 6)$ is on the graph of $y = f(-x)$. If the point $(6, 3)$ is on the graph of $y = f(-x)$ then we have $f(-(6)) = 3$; but we know nothing about $f(-6)$ so this point may or may not be on the graph of $y = f(-x)$. If the point $(6, -3)$ is on the graph of $y = f(-x)$ then we have $f(-(6)) = -3$; but we know nothing about $f(-6)$ so this point may or may not be on the graph of $y = f(-x)$. If the point $(3, -6)$ is on the graph of $y = f(-x)$ then we have $f(-(3)) = -6$; but we know nothing about $f(-3)$ so this point may or may not be on the graph of $y = f(-x)$. Therefore,

$\boxed{\text{point } (-3, 6) \text{ is the only point which must be on the graph of } y = f(-x)}$.

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Page 103 Number 34

Page 103 Number 34. If $(4, 2)$ is a point on the graph of $y = f(x)$, which of the following points must be on the graph of $y = f(2x)$? **(a)** $(4, 1)$, **(b)** $(8, 2)$, **(c)** $(2, 2)$, **(d)** $(4, 4)$.

Solution. Since $(4, 2)$ is on the graph of f, then $2 = f(4)$ and so $2 = f(2(2))$ and $\boxed{\text{the point } (2, 2) \text{ is on the graph of } y = f(2x)}$. If the point $(4, 1)$ is on the graph of $y = f(2x)$ then we have $f(2(4)) = f(8) = 1$; but we know nothing about $f(8)$ so this point may or may not be on the graph of $y = f(2x)$. If the point $(8, 2)$ is on the graph of $y = f(2x)$ then we have $f(2(8)) = f(16) = 2$; but we know nothing about $f(16)$ so this point may or may not be on the graph of $y = f(2x)$. If the point $(4, 4)$ is on the graph of $y = f(2x)$ then we have $f(2(4)) = f(8) = 4$; but we know nothing about $f(8)$ so this point may or may not be on the graph of $y = f(2x)$. □

Page 104 Number 36

Page 104 Number 36. Suppose that the x-intercepts of the graph of $y = f(x)$ are -8 and 1. **(a)** What are the x-intercepts of the graph of $y = f(x+4)$? **(b)** What are the x-intercepts of the graph of $y = f(x-3)$? **(c)** What are the x-intercepts of the graph of $y = 2f(x)$? **(d)** What are the x-intercepts of the graph of $y = f(-x)$?

Solution. The x-intercepts of the graph of $y = f(x)$ are the solutions to the equation $f(x) = 0$, so we must have $f(-8) = f(1) = 0$.

(a) The x-intercepts of the graph of $y = f(x+4)$ are the solutions to the equation $f(x+4) = 0$, so we must have $x + 4 = -8$ or $x + 4 = 1$. That is, the x-intercepts of $y = f(x+4)$ are $\boxed{x = -12 \text{ and } x = -3}$. □

(b) The x-intercepts of the graph of $y = f(x-3)$ are the solutions to the equation $f(x-3) = 0$, so we must have $x - 3 = -8$ or $x - 3 = 1$. That is, the x-intercepts of $y = f(x-3)$ are $\boxed{x = -5 \text{ and } x = 4}$. □

Page 104 Number 36 (continued)

Page 104 Number 36. Suppose that the x-intercepts of the graph of $y = f(x)$ are -8 and 1. **(a)** What are the x-intercepts of the graph of $y = f(x+4)$? **(b)** What are the x-intercepts of the graph of $y = f(x-3)$? **(c)** What are the x-intercepts of the graph of $y = 2f(x)$? **(d)** What are the x-intercepts of the graph of $y = f(-x)$?

Solution (continued). (c) The x-intercepts of the graph of $y = 2f(x)$ are the solutions to the equation $2f(x) = 0$ or $f(x) = 0$. So the x-intercepts of $y = 2f(x)$ are also $\boxed{x = -8 \text{ and } x = 1}$. □

(d) The x-intercepts of the graph of $y = f(-x)$ are the solutions to the equation $f(-x) = 0$, so we must have $-x = -8$ or $-x = 1$. That is, the x-intercepts of $y = f(-x)$ are $\boxed{x = 8 \text{ and } x = -1}$. □

Page 104 Number 46

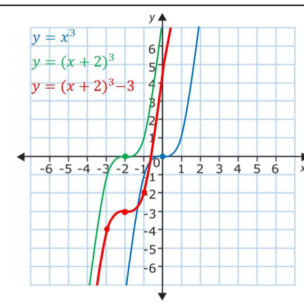
Page 104 Number 46. Consider $f(x) = (x+2)^3 - 3$. Graph f using the techniques of shifting, compressing, stretching, and/or reflecting. Start with the graph of the basic function ($y = x^3$ here), and show all stages. Be sure to show at least three key points. Find the domain and the range of f.

Solution. Starting with the graph of $y = x^3$ and replace x with $x + 2 = x - (-2)$ to get $y = (x+2)^3$; this is a horizontal shift of $y = x^3$ to the left by 2 units. Next we subtract 3 from the graph of $y = (x+2)^3$ to get $y = (x+2)^3 - 3$; this is a vertical shift down by 3 units of $y = (x+2)^3$.

So the graph of $y = (x+2)^3 - 3$ results from the graph of $y = x^3$ by $\boxed{\text{(1) a horizontal shift to the left by 2 units}}$, and then $\boxed{\text{(2) a vertical shift down by 3 units}}$.

Page 104 Number 46 (continued)

Solution. Three key points on the graph of $y = (x+2)^3 - 3$ are $\boxed{(-3,-4),\ (-2,-3), \text{ and } (-1,-2)}$ (which correspond to the points $(-1,-1)$, $(0,0)$, and $(1,1)$, respectively, on the graph of $y = x^3$ because $(-1+(-2), -1+(-3)) = (-3,-4)$, $(0+(-2), 0+(-3)) = (-2,-3)$, and $(1+(-2), 1+(-3)) = (-1,-2)$); these points are marked on the graph below. We see from the graph that, like $y = x^3$, the $\boxed{\text{domain and range are both all real numbers } (-\infty,\infty) = \mathbb{R}}$.



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Page 104 Number 60

Page 104 Number 60. Consider $g(x) = 4\sqrt{2-x}$. Graph g using the techniques of shifting, compressing, stretching, and/or reflecting. Start with the graph of the basic function ($y = \sqrt{x}$ here), and show all stages. Be sure to show at least three key points. Find the domain and the range of g.

Solution. Starting with the graph of $y = \sqrt{x}$ and replace x with $-x$ to get $y = \sqrt{-x}$; this is a reflection about the y-axis of the graph of $y = \sqrt{x}$. Next we replace x in $y = \sqrt{-x}$ with $x - 2$ to get $y = \sqrt{-(x-2)} = \sqrt{2-x}$; this is a horizontal shift to the right by 2 units of $y = \sqrt{-x}$. Finally, we consider the graph $y = 4\sqrt{2-x}$ which is a vertical stretch of $y = \sqrt{2-x}$ by a factor of 4.
So the graph of $y = 4\sqrt{2-x}$ results from the graph of $y = \sqrt{x}$ by

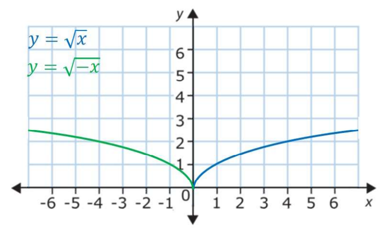
(1) a reflection about the y-axis,

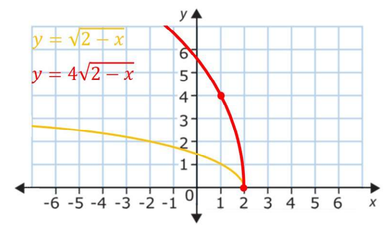
(2) a horizontal shift to the right by 2 units, and then

(3) a vertical stretch be a factor of 4.

Page 104 Number 60 (continued)

Solution. Three key points on the graph of $y = 4\sqrt{2-x}$ are $(2,0)$, $(1,4)$, and $(-2,8)$ (which correspond to the points $(0,0)$, $(1,1)$, and $(4,2)$, respectively, on the graph of $y = \sqrt{x}$ because $(-((0)-2), 4\sqrt{2-(2)}) = (2,0)$, $(-((1)-2), 4\sqrt{2-(1)}) = (1,4)$, and $(-((4)-2), 4\sqrt{2-(-2)}) = (-2,8)$); the points $(2,0)$ and $(1,4)$ are marked on the graph below. We see from the graph that the domain is $(-\infty, 2]$ and range is $[0, \infty)$.





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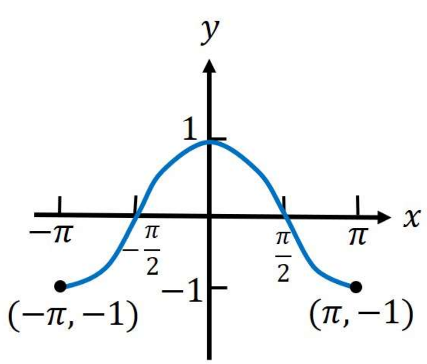
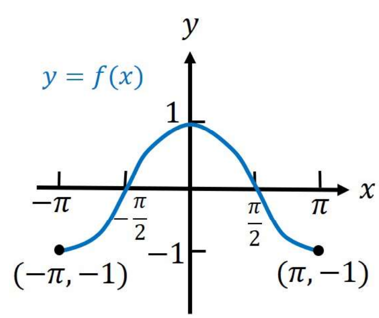
Page 104 Number 66

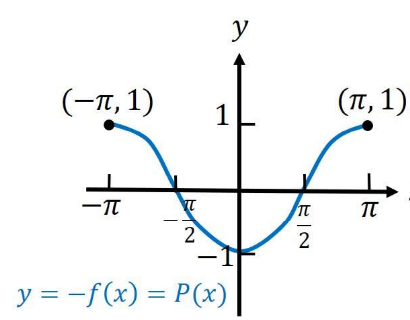
Page 104 Number 66. The graph of a function f is illustrated. Use the graph of f as the first step toward graphing each of the following functions: **(c)** $P(x) = -f(x)$, **(d)** $H(x) = f(x+1) - 2$, **(f)** $G(x) = f(-x)$, **(g)** $h(x) = f(2x)$.



Page 104 Number 66 (continued 1)

Solution. (c) To get $P(x) = -f(x)$ from $f(x)$, we multiply $f(x)$ by -1 resulting in a reflection of the graph of $y = f(x)$ about the x-axis.





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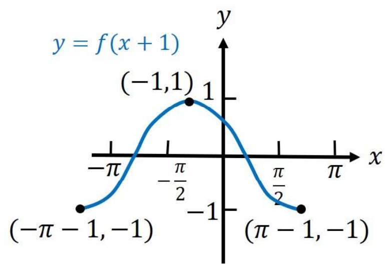
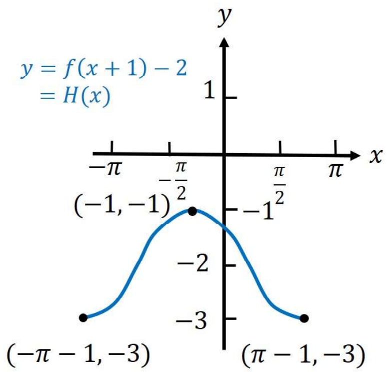
Page 104 Number 66 (continued 2)

Solution. **(d)** To get $H(x) = f(x+1) - 2$ from $f(x)$, we first replace x with $x - (-1) = x + 1$ (resulting in a horizontal shift to the left by 1 unit) and then subtract 2 from the resulting function (resulting in a vertical shift down by 2 units).





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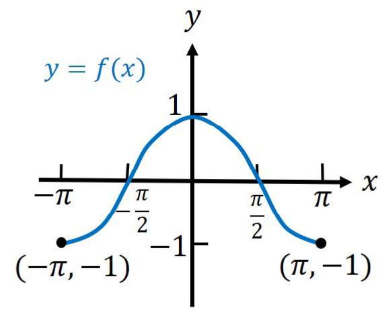
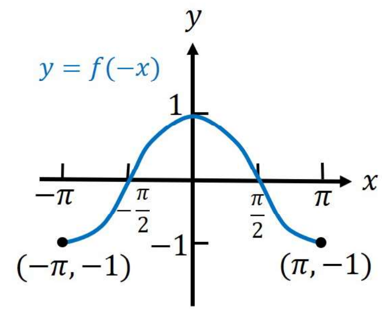
Page 104 Number 66 (continued 3)

Solution. **(f)** To get $G(x) = f(-x)$ from $f(x)$, we replace x with $-x$ resulting in a reflection of the graph of $y = f(x)$ about the y-axis.





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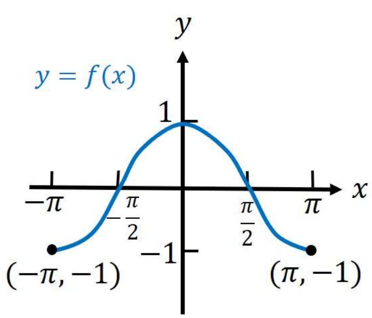
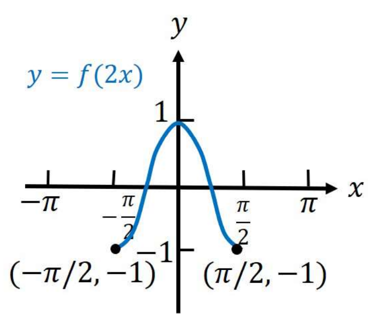
Page 104 Number 66 (continued 4)

Solution. **(g)** To get $h(x) = f(2x)$ from $f(x)$, we replace x with $2x$ resulting in a horizontal compression.





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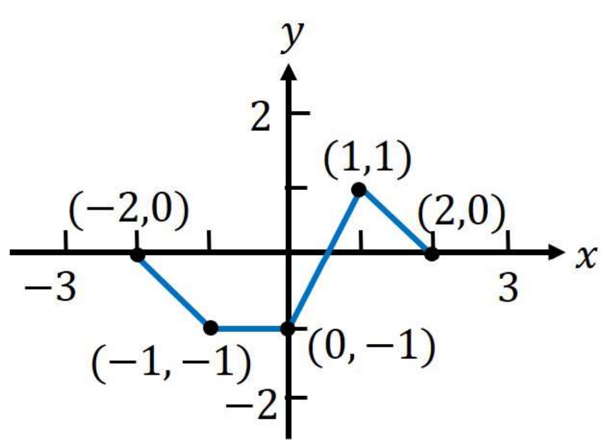
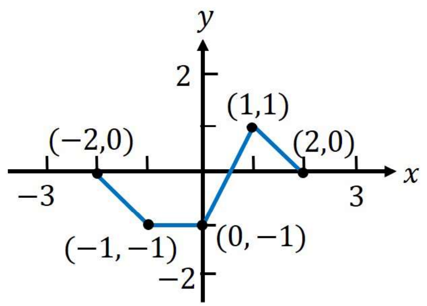
Page 105 Number 76

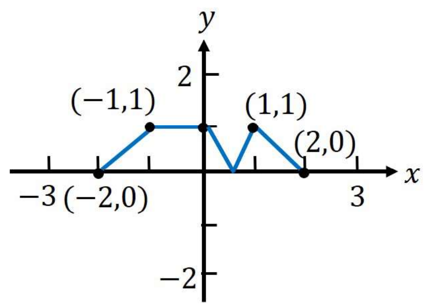
Page 105 Number 76. The graph of a function f is illustrated in the figure. **(a)** Draw the graph of $y = |f(x)|$. **(b)** Draw the graph of $y = f(|x|)$.



Page 105 Number 76 (continued 1)

Solution. (a) To graph $y = |f(x)|$, we simply get the same points as on the graph of $y = f(x)$ when $y = f(x) \geq 0$. When $y = f(x) < 0$, we get the reflection of the points on the graph of $y = f(x)$ about the x-axis (since $|f(x)| = -f(x)$ when $f(x) < 0$).





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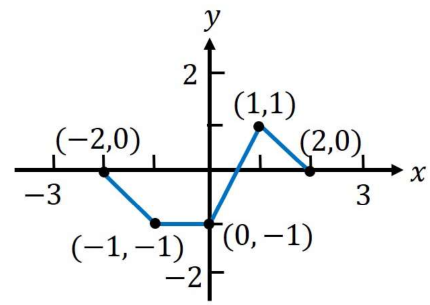
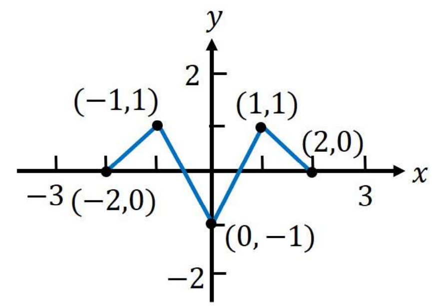
Page 105 Number 76 (continued 2)

Solution. (b) To graph $y = f(|x|)$, we simply get the same points as on the graph of $y = f(x)$ when $x \geq 0$. When $x < 0$, we get the points on the graph of $y = f(x)$ for $x > 0$ reflected about the y-axis (since $|x| = -x$ when $x < 0$).





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