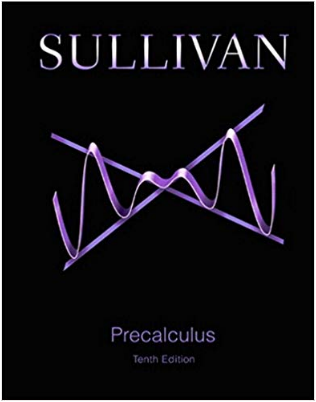
Precalculus 1 (Algebra)

Chapter 2. Functions and Their Graphs

2.6. Mathematical Models: Building Functions—Exercises, Examples, Proofs



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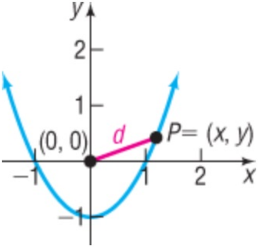

Solution. (a) From the distance formula,
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2} = \sqrt{x^2 + (x^2 - 1)^2} = \sqrt{x^2 + x^4 - 2x^2 + 1} = \boxed{\sqrt{x^4 - x^2 + 1}}$. □

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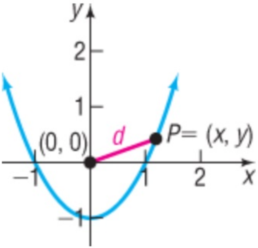

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$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2} = \sqrt{x^2 + (x^2 - 1)^2} = \sqrt{x^2 + x^4 - 2x^2 + 1} = \boxed{\sqrt{x^4 - x^2 + 1}}$. □

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Solution. **(b)** From part (a), $d = \sqrt{x^4 - x^2 + 1}$. So when $x = 0$, $\boxed{d = \sqrt{(0)^4 - (0)^2 + 1} = 1}$. □

(c) When $x = 1$, $\boxed{d = \sqrt{(1)^4 - (1)^2 + 1} = 1}$. □

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(d) When $x = \sqrt{2}/2$,

$$\boxed{d = \sqrt{(\sqrt{2}/2)^4 - (\sqrt{2}/2)^2 + 1} = \sqrt{1/4 - 1/2 + 1} = \sqrt{3/4}}.$$ □

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Page 109 Number 6

Page 109 Number 6. A right triangle has one vertex on the graph of $y = 9 - x^2$, $x > 0$, at (x, y), another at the origin, and the third on the positive x-axis at $(x, 0)$. Express the area A of the triangle as a function of x.

Solution. The picture for the problem is:

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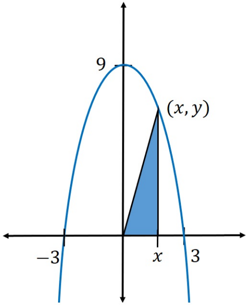

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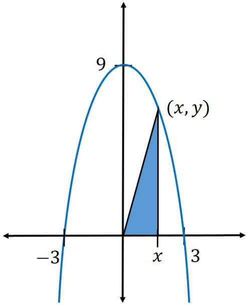

The base of the triangle is $b = x$ and the height is $h = y = 9 - x^2$. So the area is $\boxed{A = \frac{1}{2}bh = \frac{1}{2}x(9 - x^2)}$. □

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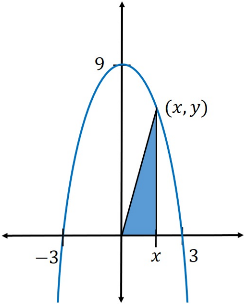

The base of the triangle is $b = x$ and the height is $h = y = 9 - x^2$. So the area is

$$\boxed{A = \frac{1}{2}bh = \frac{1}{2}x(9 - x^2)}.$$

□

Page 109 Number 10

Page 109 Number 10. A circle of radius r is inscribed in a square. See the figure.

(a) Express the area A of the square as a function of the radius r of the circle. **(b)** Express the perimeter p of the square as a function of r.

Solution. (a) Since the length ℓ of an edge of the square is twice the radius of the circle $2r$, then $\ell = 2r$. Since the area of a square with side of length ℓ is $A = \ell^2$, then $\boxed{A = (2r)^2 = 4r^2}$. □

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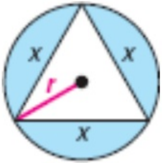
(a) Express the area A of the square as a function of the radius r of the circle. **(b)** Express the perimeter p of the square as a function of r.

Solution (continued). (b) As observed in part (a), the length ℓ of the edge of the square $\ell = 2r$. Since the perimeter of a square with side of length ℓ is $p = 4\ell$, then $\boxed{p = 4(2r) = 8r}$. □

Page 110 Number 16

Page 110 Number 16. An equilateral triangle is inscribed in a circle of radius r. See the figure. Express the circumference C of the circle as a function of the length x of a side of the triangle.



Solution. First, we need a relationship between x and r.

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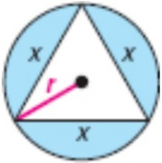

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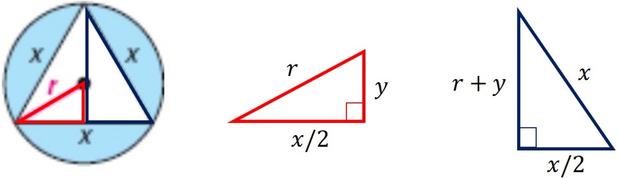
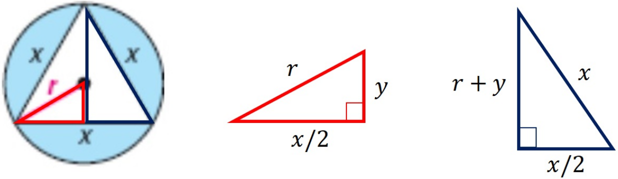
Page 110 Number 16 (continued)



Solution. We introduce the two right triangles in the figure. So by the Pythagorean Theorem we have for the red triangle that $y^2 + (x/2)^2 = r^2$ or $x^2/4 = r^2 - y^2$, and for the blue triangle that $(r + y)^2 + (x/2)^2 = x^2$ or $(r + y)^2 = x^2 - x^2/4 = 3x^2/4$.

Page 110 Number 16 (continued)



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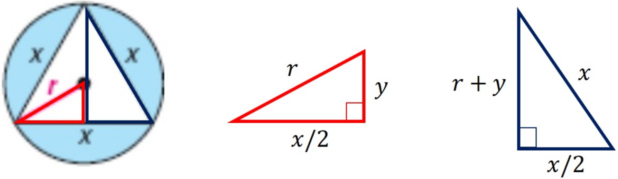
Page 110 Number 16 (continued)



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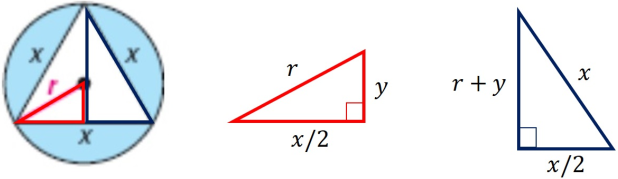
Page 110 Number 16 (continued)



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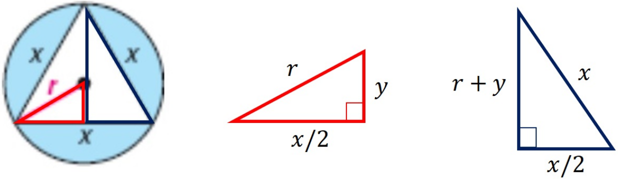

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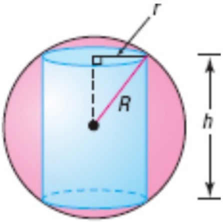
Page 110 Number 20

Page 110 Number 20. Inscribe a right circular cylinder of height h and radius r in a sphere of fixed radius R. See the illustration. Express the volume V of the cylinder as a function of h.



Sphere

Solution. The triangle in the illustration has height $h/2$, so by the Pythagorean Theorem we have $(h/2)^2 + r^2 = R^2$. The volume of a cylinder with height h and radius r is $V = \pi r^2 h$. Since $(h/2)^2 + r^2 = R^2$ then $r^2 = R^2 - h^2/4$ and so we have $\boxed{V = \pi(R^2 - h^2/4)h}$. □

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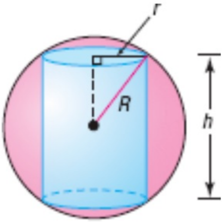

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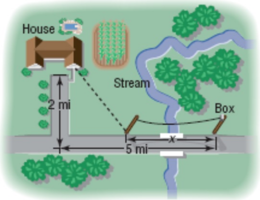
Solution. The triangle in the illustration has height $h/2$, so by the Pythagorean Theorem we have $(h/2)^2 + r^2 = R^2$. The volume of a cylinder with height h and radius r is $V = \pi r^2 h$. Since $(h/2)^2 + r^2 = R^2$ then $r^2 = R^2 - h^2/4$ and so we have $\boxed{V = \pi(R^2 - h^2/4)h}$. □

Page 110 Number 22

Page 110 Number 22. MetroMedia Cable is asked to provide service to a customer whose house is located 2 miles from the road along which the cable is buried. The nearest connection box for the cable is located 5 miles down the road. See the figure.



(a) If the installation cost is \$500 per mile along the road and \$700 per mile off the road, build a model that expresses the total cost c of installation as a function of the distance x (in miles) from the connection box to the point where the cable installation turns off the road. Find the domain of $c = c(x)$. **(b)** Compute the cost if $x = 1$ mile. **(c)** Compute the cost if $x = 3$ miles.

Page 110 Number 22 (continued 1)

Page 110 Number 22. (a) If the installation cost is \$500 per mile along the road and \$700 per mile off the road, build a model that expresses the total cost C of installation as a function of the distance x (in miles) from the connection box to the point where the cable installation turns off the road. Find the domain of $c = c(x)$.

Solution. (a) We let y be the distance from the house to the point where the cable installation turns off the road. We then have a right triangle with side of length y as follows:

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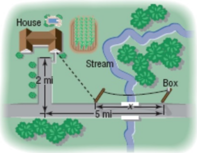

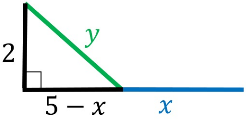

By the Pythagorean Theorem we have $(2)^2 + (5-x)^2 = y^2$ or (since y is a distance and hence positive)
$y = \sqrt{4 + (5-x)^2} = \sqrt{4 + (25 - 10x + x^2)} = \sqrt{29 - 10x + x^2}$.

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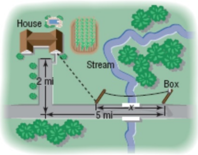

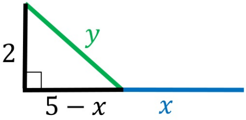

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Page 110 Number 22 (continued 2)

Page 110 Number 22. (b) Compute the cost if $x = 1$ mile. **(c)** Compute the cost if $x = 3$ miles.

Solution (continued). $\ldots y = \sqrt{29 - 10x + x^2}$. In terms of x and y, the cost is $c = 500x + 700y$. So $\boxed{c(x) = 500x + 700\sqrt{29 - 10x + x^2}}$. □

(b) With $x = 1$ we have
$c(1) = 500(1) + 700\sqrt{29 - 10(1) + (1)^2} = 500 + 700\sqrt{20} \approx$ $\boxed{3630.50 \text{ dollars}}$. □

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(c) With $x = 3$ we have
$c(3) = 500(3) + 700\sqrt{29 - 10(3) + (3)^2} = 1500 + 700\sqrt{8} \approx$ $\boxed{3479.90 \text{ dollars}}$. □

Page 110 Number 22 (continued 2)

Page 110 Number 22. **(b)** Compute the cost if $x = 1$ mile. **(c)** Compute the cost if $x = 3$ miles.

Solution (continued). $\ldots y = \sqrt{29 - 10x + x^2}$. In terms of x and y, the cost is $c = 500x + 700y$. So $\boxed{c(x) = 500x + 700\sqrt{29 - 10x + x^2}}$. □

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