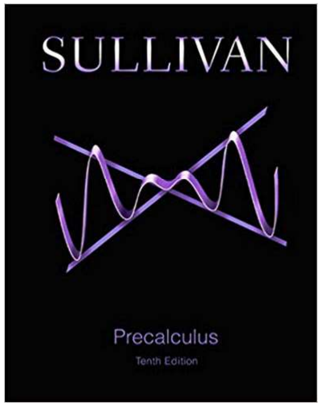
Precalculus 1 (Algebra)

Chapter 3. Linear and Quadratic Functions

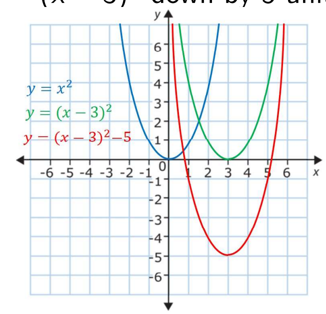
3.3. Quadratic Functions and Their Properties—Exercises, Examples, Proofs



Page 145 Number 24

Page 145 number 24. Graph the function $f(x) = (x-3)^2 - 5$ by starting with the graph of $y = x^2$ and using transformations (shifting, compressing, stretching, and/or reflecting).

Solution. First we replace x in $y = x^2$ with $x - (3)$ which results in a horizontal shift of the graph of $y = x^2$ to the right by 3 units to give the graph of $y = (x-3)^2$. Next, we subtract 5 from $(x-3)^2$ which results in shifting the graph of $y = (x-3)^2$ down by 5 units to give the graph of $f(x) = (x-3)^2 - 5$.



□

Theorem 3.3.A

Theorem 3.3.A. The x-intercepts of a quadratic function satisfy:

(1) If the discriminant $b^2 - 4ac > 0$, the graph of $f(x) = ax^2 + bx + c$ has two distinct x-intercepts and so will cross the x-axis in two places.

(2) If the discriminant $b^2 - 4ac = 0$, the graph of $f(x) = ax^2 + bx + c$ has one x-intercept and touches the x-axis at its vertex.

(3) If the discriminant $b^2 - 4ac < 0$, the graph of $f(x) = ax^2 + bx + c$ has no x-intercept and so will not cross or touch the x-axis.

Proof. We find an x-intercept by setting $f(x) = ax^2 + bx + c \equiv 0$. Solving this equation gives $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ by the quadratic formula. Since we only consider real number solutions, there are no solutions when $b^2 - 4ac < 0$ (Case 3).

Theorem 3.3.A (continued)

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(3) If the discriminant $b^2 - 4ac < 0$, the graph of $f(x) = ax^2 + bx + c$ has no x-intercept and so will not cross or touch the x-axis.

Proof (continued). When $b^2 - 4ac = 0$ there is only solution, namely $x = -\dfrac{b}{2a}$, and since the graph of f is a parabola with a vertical axis, the graph touches the x-axis at its vertex (Case 2). When $b^2 - 4ac > 0$ the quadratic formula gives two solutions (Case 1). □

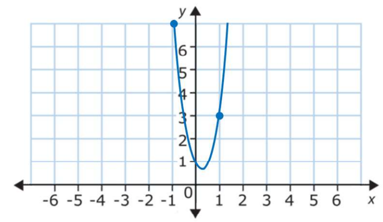
Page 146 Number 42

Page 146 Number 42. Consider $f(x) = 4x^2 - 2x + 1$. **(a)** Graph f by determining whether its graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any. **(b)** Determine the domain and the range of the function. **(c)** Determine where the function is increasing and where it is decreasing.

Solution. **(a)** For $f(x) = 4x^2 - 2x + 1$ we have $a = 4$, $b = -2$, and $c = 1$ so, since $a = 4 > 0$, the graph $\boxed{\text{opens up}}$. The vertex is $(-b/(2a), f(-b/(2a))$ where $-b/(2a) = -(-2)/(2(4)) = 1/4$, so the $\boxed{\text{vertex is } (1/4, f(1/4)) = (1/4, 4(1/4)^2 - 2(1/4) + 1) = (1/4, 3/4)}$. The axis of symmetry is the vertical line $x = -b/(2a)$ so the $\boxed{\text{axis of symmetry is } x = 1/4}$. Notice that the discriminant is $b^2 - 4ac = (-2)^2 - 4(4)(1) = -12 < 0$ so the graph of $y = f(x)$ has $\boxed{\text{no } x\text{-intercepts}}$. For the y-intercept, we set $x = 0$ to get $f(0) = 4(0)^2 - 2(0) + 1 = 1$ and so the $\boxed{y\text{-intercept is } 1}$.

Page 146 Number 42 (continued)

Solution (continued). Notice that $f(-1) = 4(-1)^2 - 2(-1) + 1 = 7$ and $f(1) = 4(1)^2 - 2(1) + 1 = 3$, so the points $(-1, 7)$ and $(1, 3)$ are on the graph. So the graph is:

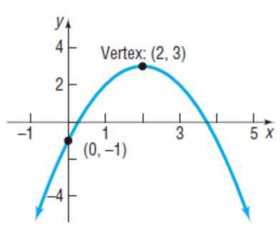


(b) As with any quadratic function, f has $\boxed{\text{domain } (-\infty, \infty) = \mathbb{R}}$. From the graph we see that the $\boxed{\text{range is } [3/4, \infty)}$. □

(c) We see from the graph that f is $\boxed{\text{decreasing on } (-\infty, 1/4)}$ and $\boxed{\text{increasing on } (1/4, \infty)}$. □

Page 146 Number 52

Page 146 Number 52. Determine the quadratic function whose graph is given:



Solution. Let the quadratic function be $f(x) = ax^2 + bx + c$. The y-intercept is $f(0) = a(0)^2 + b(0) + c = c = -1$, so $c = -1$. The axis of symmetry is $x = -b/(2a) = 2$, so $-b = 4a$ or $b = -4a$ and, in terms of a, $f(x) = ax^2 + (-4a)x - 1$. Since $f(2) = 3$, we must have $f(2) = a(2)^2 - 4a(2) - 1 = 4a - 8a - 1 = -4a - 1 = 3$ or $a = -1$, from which $b = -4(-1) = 4$. Therefore, $\boxed{f(x) = -x^2 + 4x - 1}$. □

Page 146 Number 58

Page 146 Number 58. Determine, without graphing, whether the quadratic function $f(x) = 4x^2 - 8x + 3$ has a maximum or a minimum value, and then find the value.

Solution. Let the quadratic function be $f(x) = ax^2 + bx + c$. Since $a = 4 > 0$ then the graph of $y = f(x)$ opens up and $\boxed{f \text{ has an absolute minimum}}$ at its vertex. The vertex has x-coordinate $-b/(2a) = -(-8)/(2(4)) = 1$. So the $\boxed{\text{absolute minimum of } f \text{ is } f(1) = 4(1)^2 - 8(1) + 3 = -1}$. □

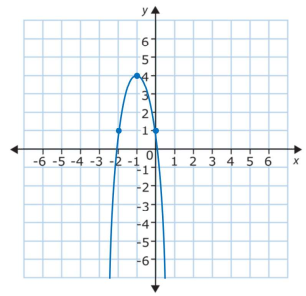
Page 146 Number 68

Page 146 Number 68. Consider $h(x) = -3(x+1)^2 + 4$. **(a)** Graph h. **(b)** Determine the domain and the range of h. **(c)** Determine where h is increasing and where it is decreasing.

Solution. **(a)** Since h is in the form $h(x) = a(x-h)^2 + k$ where $a = -3$, $h = -1$, and $k = 4$ then the vertex of the graph is $(h, k) = (-1, 4)$ and the graph opens down since $a = -3 < 0$ (see Note 3.3.A). The axis of symmetry is the vertical line $x = -1$ through the vertex. The y-intercept is $h(0) = -3((0)+1)^2 + 4 = 1$. Also, $h(-2) = -3((-2)+1)^2 + 4 = 1$. So the graph is:

Page 146 Number 68 (continued)

Solution (continued).



(b) As with any quadratic function, h has $\boxed{\text{domain } (-\infty, \infty) = \mathbb{R}}$. From the graph we see that the $\boxed{\text{range is } (-\infty, 4]}$. □

(c) We see from the graph that f is $\boxed{\text{increasing on } (-\infty, -1)}$ and $\boxed{\text{decreasing on } (-1, \infty)}$. □

Page 146 Number 80

Page 146 Number 80. **(a)** Find a quadratic function whose x-intercepts are -5 and 3 with $a = 1$; $a = 2$; $a = -2$; $a = 5$. **(b)** How does the value of a affect the intercepts? **(c)** How does the value of a affect the axis of symmetry? **(d)** How does the value of a affect the vertex? **(e)** Compare the x-coordinate of the vertex with the midpoint of the x-intercepts. What might you conclude?

Solution. **(a)** Since the x-intercepts are where $y = f(x) = 0$, then with -5 and 3 as x-intercepts we must have $x - (-5) = x + 5$ and $x - 3$ as factors of f. So f is of the form
$f(x) = a(x+5)(x-3) = a(x^2 + 2x - 15) = ax^2 + 2ax - 15a = f_a(x)$.
With $\boxed{a = 1 \text{ we have } f_1(x) = x^2 + 2x - 15}$, with
$\boxed{a = 2 \text{ we have } f_2(x) = 2x^2 + 4x - 30}$, with
$\boxed{a = -2 \text{ we have } f_{-2}(x) = -2x^2 - 4x + 30}$, and with
$\boxed{a = 5 \text{ we have } f_5(x) = 5x^2 + 10x - 75}$.

Page 146 Number 80 (continued)

Solution (continued). **(b)** If the x intercepts are -5 and 3 then f is of the form $f(x) = a(x+5)(x-3) = ax^2 + 2ax - 15a$ and a
$\boxed{\text{does not affect the } x\text{-intercepts}}$. □

(c) The axis of symmetry is the vertical line $x = -b/(2a)$ and for $f(x) = ax^2 + 2ax - 15a$ we have $b = 2a$ so that for any value of a the axis of symmetry is $x = -(2a)/(2a) = -1$ and so a
$\boxed{\text{does not affect the axis of symmetry}}$. □

(d) The vertex has coordinates $(-b/(2a), f(-b/(2a))$ and from part (c), $-b/(2a) = -1$. We have $f(-1) = a(-1)^2 + 2a(-1) - 15(a) = -16a$ so the vertex is $(-1, -16a)$. So
$\boxed{\text{as } a \text{ increases, the } y\text{-coordinate of the vertex decreases}}$. □

(e) The x-coordinate of the vertex is -1 and the midpoint of the intercepts is $((-5) + (3))/2 = -2/2 = -1$. So these are equal. We conclude that the x-coordinate of the vertex and the midpoint of the x-intercepts $\boxed{\text{are, in general, equal}}$. □

Page 146 Number 92

Page 146 Number 92. In the United States, the birth rate B of unmarried women (births per 1000 unmarried women) for women whose age is a is modeled by the function $B(a) = -0.30a^2 + 16.26a - 158.90$.
(a) What is the age of unmarried women with the highest birth rate?
(b) What is the highest birth rate of unmarried women? **(c)** Evaluate and interpret $B(40)$.

Solution. (a) First, function B is a quadratic function. Since the coefficient of the square of the variable is -0.30 (negative), then the graph of B opens down and B has an absolute maximum at the vertex. The vertex is at $(a, B(a))$ where $a = -(16.26)/(2(-0.30)) = 27.1$, so the age of unmarried women with the highest birth rate is $\boxed{a = 27.1 \text{ years}}$. □

(b) The highest birth rate is the function value at the vertex and so is $B(27.1) = -0.30(27.1)^2 + 16.26(27.1) - 158.90 =$ $\boxed{61.423 \text{ births per 1000 unmarried women}}$. □

Page 146 Number 92 (continued)

Page 146 Number 92. In the United States, the birth rate B of unmarried women (births per 1000 unmarried women) for women whose age is modeled by the function $B(a) = -0.30a^2 + 16.26a - 158.90$.
(a) What is the age of unmarried women with the highest birth rate?
(b) What is the highest birth rate of unmarried women? **(c)** Evaluate and interpret $B(40)$.

Solution (continued). (c) We have $B(40) = -0.30(40)^2 + 16.26(40) - 158.90 =$ $\boxed{11.5 \text{ births per 1000 unmarried women}}$. We interpret this as: Of 1000 unmarried women of age 40 in the United States, 11.5 will give birth (in a given year on average). □