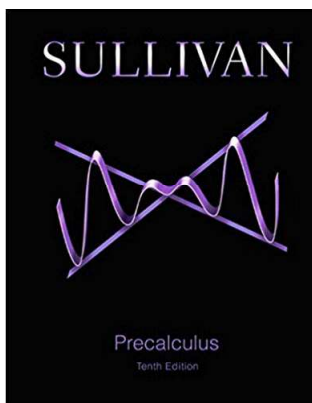


Precalculus 1 (Algebra)

Chapter 3. Linear and Quadratic Functions

3.4. Build Quadratic Models from Verbal Descriptions and from Data—Exercises, Examples, Proofs



()

Precalculus 1 (Algebra)

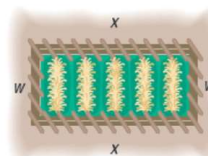
September 8, 2021 1 / 11

Example 3.4.2. Maximizing the Area Enclosed by a Fence

Example 3.4.2

Example 3.4.2. A farmer has 2000 yards of fence to enclose a rectangular field. What are the dimensions of the rectangle that encloses the most area?

Solution. We introduce variables x and w where x is the length of the rectangular field and w is the width of the field:



Then the perimeter of field is $2x + 2w$ and this equals the amount of fence so that $2x + 2w = 2000$ yards. Since $2x + 2w = 2000$, then $2w = 2000 - 2x$ or $w = 1000 - x$. Now the area of the field is $A = xw$ so we can write the area as a function of x alone as $A(x) = x(1000 - x) = -x^2 + 1000x$.

()

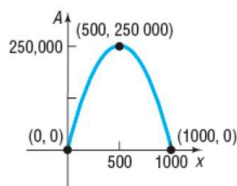
Precalculus 1 (Algebra)

September 8, 2021 3 / 11

Example 3.4.2. Maximizing the Area Enclosed by a Fence

Example 3.4.2 (continued)

Solution (continued). ... $A(x) = x(1000 - x) = -x^2 + 1000x$. Hence, A is a quadratic function $ax^2 + bx + c$ where $a = -1$, $b = 1000$, and $c = 0$. So the graph of A opens down and A has an absolute maximum at its vertex. The x -coordinate of the vertex is $-b/(2a) = -(1000)/(2(-1)) = 500$ and so the area is a maximum when $x = 500$ yards and $w = 1000 - (500) = 500$ yards; that is, when the length is 500 yards and the width is 500 yards. Notice that the maximum area is $A(500) = -(500)^2 + 1000(500) = 250,000$ yards². The graph of A is:



Page 150 Figure 26



()

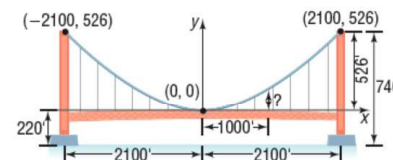
Precalculus 1 (Algebra)

September 8, 2021 4 / 11

Example 3.4.4. The Golden Gate Bridge

Example 3.4.4

Example 3.4.4. The Golden Gate Bridge, a suspension bridge, spans the entrance to San Francisco Bay. Its 746-foot-tall towers are 4200 feet apart. The bridge is suspended from two huge cables more than 3 feet in diameter; the 90-foot-wide roadway is 220 feet above the water. The cables are parabolic in shape and touch the road surface at the center of the bridge. Find the height of the cable above the road at a distance of 1000 feet from the center.



Page 152 Figure 28

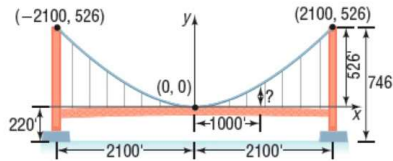
()

Precalculus 1 (Algebra)

September 8, 2021 5 / 11

Example 3.4.4

Solution. Since the cable forms a parabola, then the function describing it is a quadratic function $f(x) = ax^2 + bx + c$. We use the x -axis and y -axis introduced in the image so that the origin is at the low point on the cable (i.e., the “center”). So the vertex of the graph of f is $(0, 0)$ and the x -coordinate of the vertex is $0 = -b/(2a)$ so that $b = 0$. The y -intercept is 0 so $f(0) = a(0)^2 + b(0) + c = 0$ and $c = 0$. Therefore, $b = c = 0$ and f is of the form $f(x) = ax^2$ for some a . We are given that the points $(-2100, 526)$ and $(2100, 526)$ are on the graph of f so we use this information to find a . We must have $f(2100) = a(2100)^2 = 526$ or $a = 526/(2100)^2 \approx 0.000119$. Hence $f(x) = (0.000119)x^2$ and 1000 feet from the center (when $x = 1000$) we have the height of the cable above the road is $f(1000) = (0.000119)(1000)^2 = \boxed{119 \text{ feet}}$. \square



Page 154 Number 6

Page 154 Number 6. The price p (in dollars) and the quantity x sold of a certain product obey the demand equation $x = -20p + 500$ where $0 < p \leq 25$. **(a)** Express the revenue R as a function of x . **(b)** What is the revenue if 20 units are sold? **(c)** What quantity x maximizes revenue? What is the maximum revenue? **(d)** What price should the company charge to maximize revenue?

Solution. **(a)** At a price of p dollars each, with x units sold the revenue function is $R(x) = px$. Since $x = -20p + 500$, then $20p = 500 - x$ or $p = (500 - x)/20 = 25 - x/20$ and hence

$$R(x) = (25 - x/20)x = 25x - x^2/20 \text{ dollars.} \quad \square$$

(b) If 20 items are sold then $x = 20$ and so the revenue if 20 units are sold is $R(20) = 25(20) - (20)^2/20 = \boxed{480 \text{ dollars}}$. \square

Page 154 Number 6 (continued)

Page 154 Number 6. The price p (in dollars) and the quantity x sold of a certain product obey the demand equation $x = -20p + 500$ where $0 < p \leq 25$. **(c)** What quantity x maximizes revenue? What is the maximum revenue? **(d)** What price should the company charge to maximize revenue?

Solution (continued). **(c)** Since $R(x) = (-1/20)x^2 + 25x$ then R is a quadratic function and its graph opens down since $a = -1/20 < 0$. So R has a maximum at its vertex. Also, $b = 25$ and the vertex has x -coordinate $x = -b/(2a) = -(25)/(2(-1/20)) = 250$ so

revenue is maximized when $x = 250$. Now

$R(250) = (-1/20)(250)^2 + 25(250) = -62,500/20 + 6,250 = 3,125$ and the $\boxed{\text{maximum revenue is } R(250) = 3,125 \text{ dollars}}$.

(d) Since $p = 25 - x/20$ by part (a) then at maximum revenue, which occurs at $x = 250$ (by part (c)), we have the price

$$p = 25 - (250)/20 = 12.50 \text{ dollars.} \quad \square$$

Page 155 Number 16

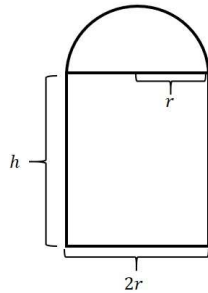
Page 155 Number 16. A Norman window has the shape of a rectangle surmounted by a semicircle of diameter equal to the width of the rectangle. See the figure. If the perimeter of the window is 20 feet, what dimensions will admit the most light (maximize the area)? **Hint:** The circumference C of a circle of radius r is $C = 2\pi r$; the area of a circle of radius r is $A = \pi r^2$.



Solution. We label the radius of the semicircle as r and the height of the rectangular part of the window as h

Page 155 Number 16 (continued 1)

Solution (continued).



Then the perimeter of the window is

$$2h + 2r + (2\pi r)/2 = 2h + 2r + \pi r = 20 \text{ feet. Hence}$$

$$h = (20 - 2r - \pi r)/2 = 10 - r - \pi r/2. \text{ The area of the window is}$$

$$A = (h)(2r) + (\pi r^2)/2 = 2rh + \pi r^2/2 = 2r(10 - r - \pi r/2) + \pi r^2/2 = 20r - 2r^2 - \pi r^2 + \pi r^2/2 = -2r^2 - \pi r^2/2 + 20r = -(2 + \pi/2)r^2 + 20r.$$

()

Page 155 Number 16 (continued 2)

Solution (continued). So $A(r) = -(2 + \pi/2)r^2 + 20r$ is a quadratic function of r and since $a = -(2 + \pi/2) < 0$, then the graph of A opens down and A has a maximum at its vertex. The vertex has r -coordinate $r = -b/(2a) = -(20)/(2(-(2 + \pi/2))) = 20/(4 + \pi)$. The corresponding value for h is

$$h = 10 - r - \pi r/2 = 10 - (20/(4 + \pi)) - \pi(20/(4 + \pi))/2$$

$$= 10 - 20/(4 + \pi) - 10\pi/(4 + \pi) = 10 - (20 + 10\pi)/(4 + \pi)$$

$$= (10(4 + \pi) - (20 + 10\pi))/(4 + \pi) = 20/(4 + \pi).$$

So the dimensions which maximize area are height $h = 20/(4 + \pi)$ feet

and width $w = 2r = 40/(4 + \pi)$ feet. □

()