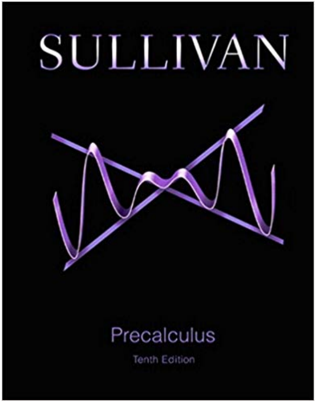
Precalculus 1 (Algebra)

Chapter 3. Linear and Quadratic Functions

3.4. Build Quadratic Models from Verbal Descriptions and from Data—Exercises, Examples, Proofs



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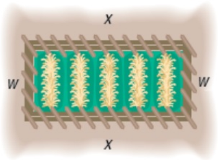
Solution. We introduce variables x and w where x is the length of the rectangular field and w is the width of the field:



Then the perimeter of field is $2x + 2w$ and this equals the amount of fence so that $2x + 2w = 2000$ yards. Since $2x + 2w = 2000$, then $2w = 2000 - 2x$ or $w = 1000 - x$.

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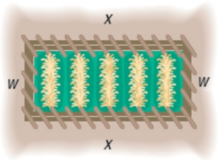

Then the perimeter of field is $2x + 2w$ and this equals the amount of fence so that $2x + 2w = 2000$ yards. Since $2x + 2w = 2000$, then $2w = 2000 - 2x$ or $w = 1000 - x$. Now the area of the field is $A = xw$ so we can write the area as a function of x alone as $A(x) = x(1000 - x) = -x^2 + 1000x$.

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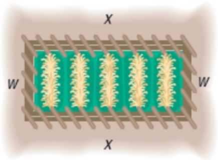

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Example 3.4.2 (continued)

Solution (continued). . . . $A(x) = x(1000 - x) = -x^2 + 1000x$. Hence, A is a quadratic function $ax^2 + bx + c$ where $a = -1$, $b = 1000$, and $c = 0$. So the graph of A opens down and A has an absolute maximum at its vertex. The x-coordinate of the vertex is $-b/(2a) = -(1000)/(2(-1)) = 500$ and so the area is a maximum when $x = 500$ yards and $w = 1000 - (500) = 500$ yards; that is, when

$\boxed{\text{the length is 500 yards and the width is 500 yards}}$.

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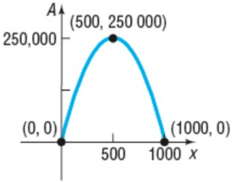

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Example 3.4.2 (continued)

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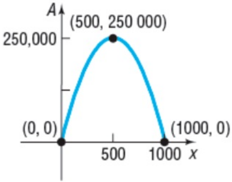

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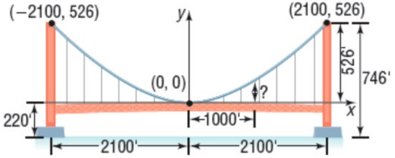
Example 3.4.4

Example 3.4.4. The Golden Gate Bridge, a suspension bridge, spans the entrance to San Francisco Bay. Its 746-foot-tall towers are 4200 feet apart. The bridge is suspended from two huge cables more than 3 feet in diameter; the 90-foot-wide roadway is 220 feet above the water. The cables are parabolic in shape and touch the road surface at the center of the bridge. Find the height of the cable above the road at a distance of 1000 feet from the center.



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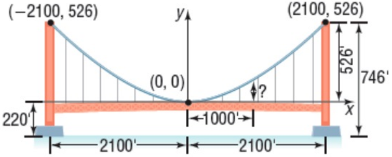

Solution. Since the cable forms a parabola, then the function describing it is a quadratic function $f(x) = ax^2 + bx + c$. We use the x-axis and y-axis introduced in the image so that the origin is at the low point on the cable (i.e., the "center"). So the vertex of the graph of f is $(0, 0)$ and the x-coordinate of the vertex is $0 = -b/(2a)$ so that $b = 0$. The y-intercept is 0 so $f(0) = a(0)^2 + b(0) + c = 0$ and $c = 0$. Therefore, $b = c = 0$ and f is of the form $f(x) = ax^2$ for some a.

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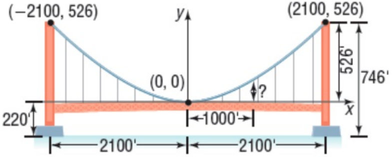

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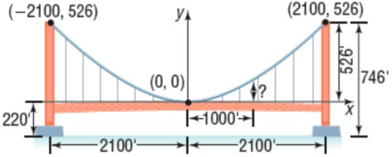

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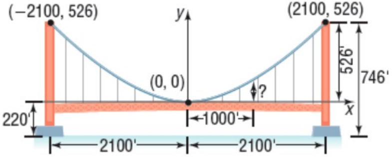

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Solution. (a) At a price of p dollars each, with x units sold the revenue function if $R(x) = px$. Since $x = -20p + 500$, then $20p = 500 - x$ or $p = (500 - x)/20 = 25 - x/20$ and hence

$$\boxed{R(x) = (25 - x/20)x = 25x - x^2/20 \text{ dollars}}.$$

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Solution (continued). (c) Since $R(x) = (-1/20)x^2 + 25x$ then R is a quadratic function and its graph opens down since $a = -1/20 < 0$. So R has a maximum at its vertex. Also, $b = 25$ and the vertex has x-coordinate $x = -b/(2a) = -(25)/(2(-1/20)) = 250$ so $\boxed{\text{revenue is maximized when } x = 250}$. Now $R(250) = (-1/20)(250)^2 + 25(250) = -62{,}500/20 + 6{,}250 = 3{,}125$ and the $\boxed{\text{maximum revenue is } R(250) = 3{,}125 \text{ dollars}}$.

(d) Since $p = 25 - x/20$ by part (a) then at maximum revenue, which occurs at $x = 250$ (by part (c)), we have the price $\boxed{p = 25 - (250)/20 = 12.50 \text{ dollars}}$. □

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Page 155 Number 16. A Norman window has the shape of a rectangle surmounted by a semicircle of diameter equal to the width of the rectangle. See the figure. If the perimeter of the window is 20 feet, what dimensions will admit the most light (maximize the area)? **Hint:** The circumference C of a circle of radius r is $C = 2\pi r$; the area of a circle of radius r is $A = \pi r^2$.

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Page 155 Number 16 (continued 1)

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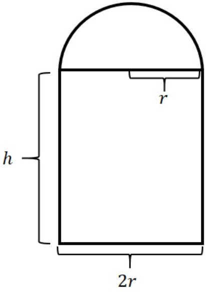

Then the perimeter of the window is
$2h + 2r + (2\pi r)/2 = 2h + 2r + \pi r = 20$ feet. Hence
$h = (20 - 2r - \pi r)/2 = 10 - r - \pi r/2$. The area of the window is
$A = (h)(2r) + (\pi r^2)/2 = 2rh + \pi r^2/2 = 2r(10 - r - \pi r/2) + \pi r^2/2 =$
$20r - 2r^2 - \pi r^2 + \pi r^2/2 = -2r^2 - \pi r^2/2 + 20r = -(2 + \pi/2)r^2 + 20r$.

Page 155 Number 16 (continued 2)

Solution (continued). So $A(r) = -(2 + \pi/2)r^2 + 20r$ is a quadratic function of r and since $a = -(2 + \pi/2) < 0$, then the graph of A opens down and A has a maximum at its vertex. The vertex has r-coordinate $r = -b/(2a) = -(20)/(2(-(2 + \pi/2))) = 20/(4 + \pi)$. The corresponding value for h is

$$h = 10 - r - \pi r/2 = 10 - (20/(4 + \pi)) - \pi(20/(4 + \pi))/2$$

$$= 10 - 20/(4 + \pi) - 10\pi/(4 + \pi) = 10 - (20 + 10\pi)/(4 + \pi)$$

$$= (10(4 + \pi) - (20 + 10\pi))/(4 + \pi) = 20/(4 + \pi).$$

So the dimensions which maximize area are $\boxed{\text{height } h = 20/(4 + \pi) \text{ feet}}$ and $\boxed{\text{width } w = 2r = 40/(4 + \pi) \text{ feet}}$. □

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