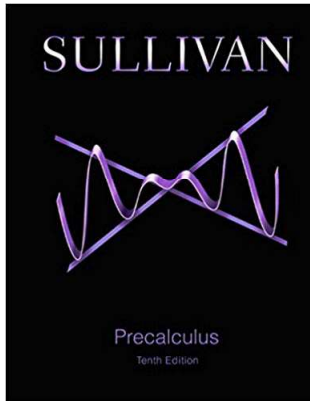


Precalculus 1 (Algebra)

Chapter 4. Polynomial and Rational Functions

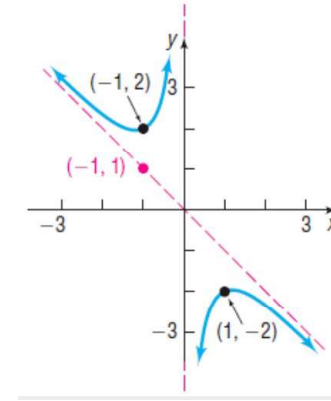
4.2. Properties of Rational Functions—Exercises, Examples, Proofs



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Page 197 Number 30

Page 197 Number 30. Use the graph shown to find **(a)** the domain and range of each function, **(b)** the intercepts, if any, **(c)** horizontal asymptotes, if any, **(d)** vertical asymptotes, if any, and **(e)** oblique asymptotes, if any.



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Page 197 Number 30 (continued)

Solution. (a) The only x -value for which the function is undefined is $x = 0$, so the domain is all real numbers except 0, $(-\infty, 0) \cup (0, \infty)$. We see that the function has a local minimum of 2 at $x = -1$ and a local maximum of -2 at $x = 1$, so the range is $(-\infty, -2] \cup [2, \infty)$.

(b) We see from the graph that the function does not intersect either axis and so has no intercepts.

(c) From the graph we see that there are no horizontal asymptotes.

(d) From the graph we see that the y -axis is a vertical asymptote. That is, the vertical asymptote is the line $x = 0$.

(e) The dashed line through the origin $(0, 0)$ and the point $(-1, 1)$ is an oblique asymptote. The slope for this line is $m = -1$ and the y -intercept is $y = 0$, so the oblique asymptote is $y = -x$.

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Page 197 Number 44

Page 197 Number 44. Consider $R(x) = \frac{x-4}{x}$. **(a)** Graph rational function $R(x)$ using transformations, **(b)** use the final graph to find the domain and range, and **(c)** use the final graph to list any vertical, horizontal, or oblique asymptotes.

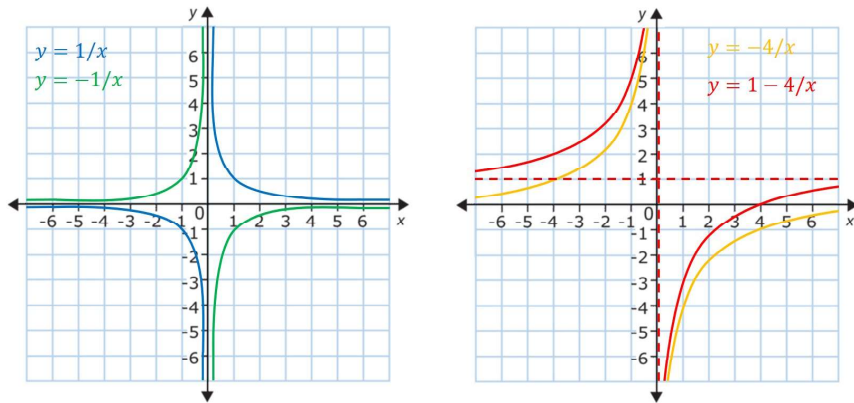
Solution. Notice that $R(x) = \frac{x-4}{x} = 1 - 4\frac{1}{x}$. The graph of $y = 1/x$ is in our library of functions.

(a) Starting with the graph of $y = 1/x$, multiply y by -1 to get $y = -1/x$; this is a reflection about the x -axis of the graph of $y = 1/x$. Next we multiply function values by 4 to get $y = -4/x$; this is a vertical stretch by a factor of 4 of the graph of $y = -1/x$. Finally, we add 1 to function values to get $y = -4/x + 1 = 1 - 4/x$; this is a vertical shift up by 1 unit of the graph of $y = -4/x$. So the graph of $R(x) = 1 - 4/x$ results from the graph of $y = 1/x$ by (1) a reflection about the x -axis, (2) a vertical stretch by a factor of 4, and (3) a vertical shift up by 1 unit.

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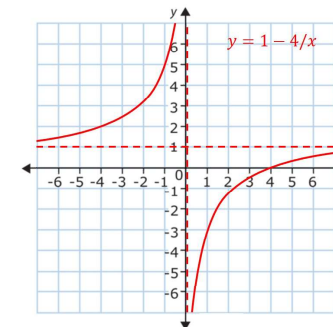
Page 197 Number 44 (continued 1)

Solution (continued). The resulting graph is:



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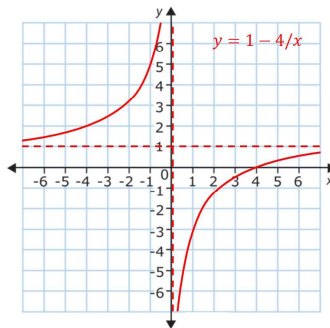
Page 197 Number 44 (continued 2)



Solution (continued). (b) We see from the graph that the domain of $R(x) = 1 - 4\frac{1}{x}$ is all real x except $x = 0$ (we also see this from the formula for R). That is, the domain of R is $(-\infty, 0) \cup (0, \infty)$. We see from the graph of R that the range is all real y except $y = 1$. That is, the range of R is $(-\infty, 1) \cup (1, \infty)$. \square

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Page 197 Number 44 (continued 3)



Solution (continued). (c) We see from the graph that R has a horizontal asymptote of $y = 1$ and a vertical asymptote of $x = 0$. \square Notice that $x - 0 = x$ is a factor of the denominator of $R(x) = (x - 4)/x$, consistent with Theorem 4.2.A which implies that $x = 0$ is a vertical asymptote.

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Page 198 Number 48

Page 198 Number 48. Find the vertical, horizontal, and oblique asymptotes, if any, of rational function $G(x) = \frac{x^3 + 1}{x^2 - 5x - 14}$.

Solution. We have $G(x) = p(x)/q(x)$ where $p(x) = x^3 + 1$ is of degree $n = 3$ and $q(x) = x^2 - 5x - 14$ is of degree $m = 2$. Since $3 = n = m + 1 = 2 + 1$, then G has an oblique asymptote $y = ax + b$ where $ax + b$ is the quotient of $p(x)/q(x)$. By long division we have:

$$\begin{array}{r} x + 5 \\ x^2 - 5x - 14 \overline{) x^3 + 1} \\ \underline{x^3 - 5x^2 - 14x} \\ 5x^2 + 14x + 1 \\ \underline{5x^2 - 25x - 70} \\ 39x + 71 \end{array}$$

So the oblique asymptote is $y = ax + b = x + 5$. \square

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Page 198 Number 48 (continued)

Page 198 Number 48. Find the vertical, horizontal, and oblique asymptotes, if any, of rational function $G(x) = \frac{x^3 + 1}{x^2 - 5x - 14}$.

Solution (continued). Now G has an oblique asymptote and so there is no horizontal asymptote. By Theorem 2.2.A, Locating Vertical Asymptotes, for vertical asymptotes we set the denominator of G equal to 0: $x^2 - 5x - 14 = 0$. This gives $(x - 7)(x + 2) = 0$ or $x = 7$ and $x = -2$. So G has vertical asymptotes of $x = -2$ and $x = 7$. □

Page 198 Number 58

Page 198 Number 58. A rare species of insect was discovered in the Amazon Rain Forest. To protect the species, environmentalists declared the insect endangered and transplanted the insect into a protected area. The population P of the insect t months after being transplanted is $P(t) = \frac{50(1 + 0.5t)}{2 + 0.01t}$. **(a)** How many insects were discovered? In other words, what was the population when $t = 0$? **(b)** What will the population be after 5 years? **(c)** Determine the horizontal asymptote of $P(t)$. What is the largest population that the protected area can sustain?

Solution. **(a)** When $t = 0$ the population is

$$P(0) = \frac{50(1 + 0.5(0))}{2 + 0.01(0)} = \frac{50}{2} = \boxed{25 \text{ insects}}. \quad \square$$

(b) At 5 years we have $t = (5)(12) = 60$ months and the population then is $P(60) = \frac{50(1 + 0.5(60))}{2 + 0.01(60)} = \frac{50(31)}{2 + (0.6)} = \frac{1550}{2.6} = 596.15$ insects, or (rounding down) 596 insects. □

Page 198 Number 58 (continued)

Page 198 Number 58. A rare species of insect was discovered in the Amazon Rain Forest. The population P of the insect t months after being transplanted is $P(t) = \frac{50(1 + 0.5t)}{2 + 0.01t}$. **(c)** Determine the horizontal asymptote of $P(t)$. What is the largest population that the protected area can sustain?

Solution (continued). **(c)** Since the numerator and denominator of P are both degree 1 polynomials, then by Note 4.2.B(2) the horizontal asymptote is a_1/b_1 where a_1 is the leading coefficient in the numerator and b_1 is the leading coefficient in the denominator. Now

$$P(t) = \frac{50(1 + 0.5t)}{2 + 0.01t} = \frac{25t + 50}{0.01t + 2}, \text{ so } a_1 = 25, b_1 = 0.01, \text{ and}$$

$a_1/b_1 = 25/0.01 = 2500$. So the horizontal asymptote is $P = 2500$. The largest population that the protected area can sustain is the population we get when t is large and this can be arbitrarily close to the horizontal asymptote. That is, the population can be as large as 2500 insects. □