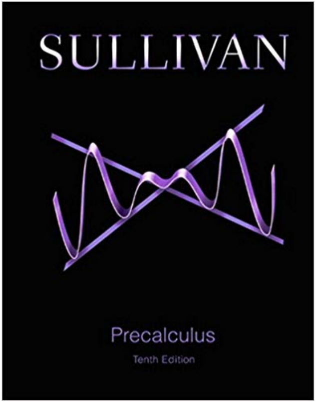
Precalculus 1 (Algebra)

Chapter 4. Polynomial and Rational Functions

4.2. Properties of Rational Functions—Exercises, Examples, Proofs



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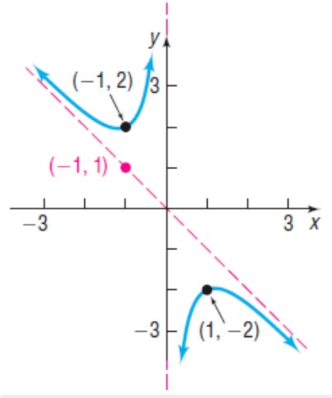
Page 197 Number 30. Use the graph shown to find **(a)** the domain and range of each function, **(b)** the intercepts, if any, **(c)** horizontal asymptotes, if any, **(d)** vertical asymptotes, if any, and **(e)** oblique asymptotes, if any.



Page 197 Number 30 (continued)

Solution. (a) The only x-value for which the function is undefined is $x = 0$, so the domain is all $\boxed{\text{real numbers except 0, } (-\infty, 0) \cup (0, \infty)}$. We see that the function has a local minimum of 2 at $x = -1$ and a local maximum of -2 at $x = 1$, so the $\boxed{\text{range is } (-\infty, -2] \cup [2, \infty)}$. □

(b) We see from the graph that the function does not intersect either axis and so has $\boxed{\text{no intercepts}}$. □

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(d) From the graph we see that the y-axis is a vertical asymptote. That is, the $\boxed{\text{vertical asymptote is the line } x = 0}$. □

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(e) The dashed line through the origin $(0, 0)$ and the point $(-1, 1)$ is an oblique asymptote. The slope for this line is $m = -1$ and the y-intercept is $y = 0$, so the $\boxed{\text{oblique asymptote is } y = -x}$. □

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Page 197 Number 44

Page 197 Number 44. Consider $R(x) = \dfrac{x-4}{x}$. **(a)** Graph rational function $R(x)$ using transformations, (b) use the final graph to find the domain and range, and (c) use the final graph to list any vertical, horizontal, or oblique asymptotes.

Solution. Notice that $R(x) = \dfrac{x-4}{x} = 1 - 4\dfrac{1}{x}$. The graph of $y = 1/x$ is in our library of functions.

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(a) Starting with the graph of $y = 1/x$, multiply y by -1 to get $y = -1/x$; this is a reflection about the x-axis of the graph of $y = 1/x$. Next we multiply function values by 4 to get $y = -4/x$; this is a vertical stretch by a factor of 4 of the graph of $y = -1/x$. Finally, we add 1 to function values to get $y = -4/x + 1 = 1 - 4/x$; this is a vertical shift up by 1 unit of the graph of $y = -4/x$.

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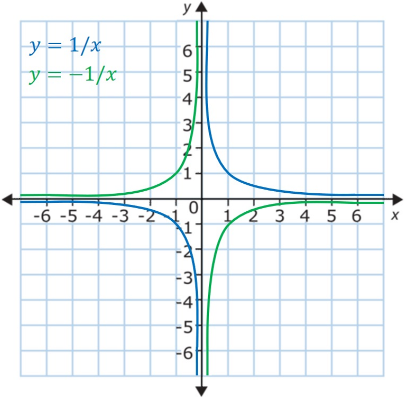
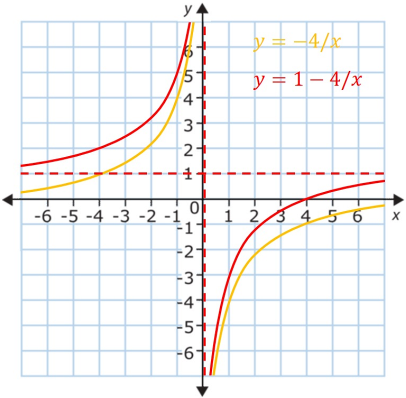
Solution (continued). The resulting graph is:





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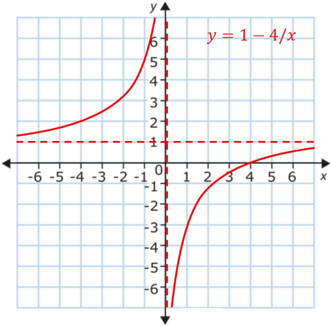

Solution (continued). (b) We see from the graph that the domain of $R(x) = 1 - 4\frac{1}{x}$ is all real x except $x = 0$ (we also see this from the formula for R). That is, the $\boxed{\text{domain of } R \text{ is } (-\infty, 0) \cup (0, \infty)}$. We see from the graph of R that the range is all real y except $y = 1$. That is, the $\boxed{\text{range of } R \text{ is } (-\infty, 1) \cup (1, \infty)}$. □

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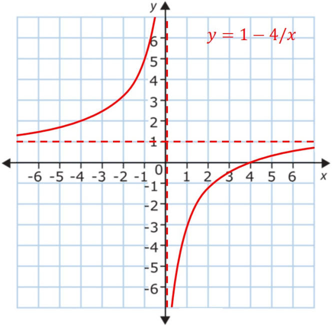

Solution (continued). (c) We see from the graph that R has a horizontal asymptote of $y = 1$ and a vertical asymptote of $x = 0$. □

Notice that $x - 0 = x$ is a factor of the denominator of $R(x) = (x - 4)/x$, consistent with Theorem 4.2.A which implies that $x = 0$ is a vertical asymptote.

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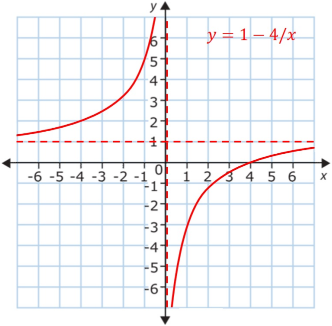

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Page 198 Number 48

Page 198 Number 48. Find the vertical, horizontal, and oblique asymptotes, if any, of rational function $G(x) = \dfrac{x^3 + 1}{x^2 - 5x - 14}$.

Solution. We have $G(x) = p(x)/q(x)$ where $p(x) = x^3 + 1$ is of degree $n = 3$ and $q(x) = x^2 - 5x - 14$ is of degree $m = 2$. Since $3 = n = m + 1 = 2 + 1$, then G has an oblique asymptote $y = ax + b$ where $ax + b$ is the quotient of $p(x)/q(x)$.

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$$\begin{array}{r|rrrr} & & & x & +\ 5 \\ \hline x^2 - 5x - 14 & x^3 & & & +\ 1 \\ & x^3 & -\ 5x^2 & -\ 14x & \\ \hline & & 5x^2 & +\ 14x & +\ 1 \\ & & 5x^2 & -\ 25x & -\ 70 \\ \hline & & & 39x & +\ 71 \end{array}$$

So the $\boxed{\text{oblique asymptote is } y = ax + b = x + 5}$. □

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Page 198 Number 58

Page 198 Number 58. A rare species of insect was discovered in the Amazon Rain Forest. To protect the species, environmentalists declared the insect endangered and transplanted the insect into a protected area. The population P of the insect t months after being transplanted is $P(t) = \dfrac{50(1+0.5t)}{2+0.01t}$. **(a)** How many insects were discovered? In other words, what was the population when $t = 0$? **(b)** What will the population be after 5 years? **(c)** Determine the horizontal asymptote of $P(t)$. What is the largest population that the protected area can sustain?

Solution. (a) When $t = 0$ the population is $P(0) = \dfrac{50(1+0.5(0))}{2+0.01(0)} = \dfrac{50}{2} = \boxed{25 \text{ insects}}$. □

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(b) At 5 years we have $t = (5)(12) = 60$ months and the population then is $P(60) = \dfrac{50(1+0.5(60))}{2+0.01(60)} = \dfrac{50(31)}{2+(0.6)} = \dfrac{1550}{2.6} = 596.15$ insects, or (rounding down) $\boxed{596 \text{ insects}}$. □

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Solution (continued). (c) Since the numerator and denominator of P are both degree 1 polynomials, then by Note 4.2.B(2) the horizontal asymptote is a_1/b_1 where a_1 is the leading coefficient in the numerator and b_1 is the leading coefficient in the denominator. Now $P(t) = \dfrac{50(1+0.5t)}{2+0.01t} = \dfrac{25t+50}{0.01t+2}$, so $a_1 = 25$, $b_1 = 0.01$, and $a_1/b_1 = 25/0.01 = 2500$. So the $\boxed{\text{horizontal asymptote is } P = 2500}$.

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