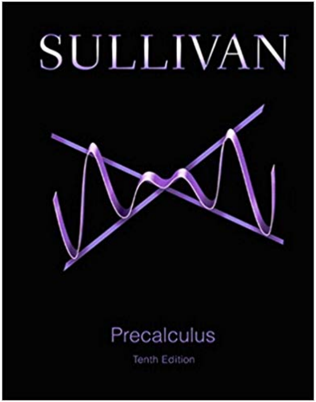
Precalculus 1 (Algebra)

Chapter 4. Polynomial and Rational Functions

4.3. The Graph of a Rational Function—Exercises, Examples, Proofs



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Page 211 Number 8. Follow Steps 1 through 7 to analyze the graph of $R(x) = \dfrac{x}{(x-1)(x+2)}$.

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In Step 5, we find horizontal or oblique asymptotes. Notice that the numerator is of degree $n = 1$ and the denominator of R is degree $m = 2$. Since $n = 1 < 2 = m$, then R has a $\boxed{\text{horizontal asymptote of } y = 0}$.

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In Step 6, we use the zeros of the numerator and denominator of R to divide the x-axis into intervals. The numerator is zero at $x = 0$ and the denominator is zero at $x = -2$ and $x = 1$, so we divide the x-axis into intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 1)$, and $(1, \infty)$.

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Page 211 Number 8 (continued 2)

Solution (continued). Consider:

Interval	$(-\infty,-2)$	$(-2,0)$	$(0,1)$	$(1,\infty)$
Test Value c	-3	-1	$1/2$	2
Value of $R(c)$	$(-3)/((-4)(-1))$	$(-1)/((-2)(1))$	$(1/2)/((-1/2)(5/2))$	$(2)/((1)(4))$
Conclusion	R negative	R is positive	R negative	R positive

So R is below the x-axis on $(-\infty,-2)\cup(0,1)$ and R is above the x-axis on $(-2,0)\cup(1,\infty)$.

Notice that since R is negative on $(-\infty,-2)$ and R has a vertical asymptote at $x=-2$, then (in the notation of Page 193 Figure 29), $\lim_{x\to-2^-} R(x)=-\infty$. Since R is positive on $(-2,0)$ and R has a vertical asymptote at $x=-2$, then $\lim_{x\to-2^+} f(x)=\infty$. Since R is negative on $(0,1)$ and R has a vertical asymptote at $x=1$, then $\lim_{x\to1^-} R(x)=-\infty$. Since R is positive on $(1,\infty)$ and R has a vertical asymptote at $x=1$, then $\lim_{x\to1^+} f(x)=\infty$ (these observations give the behaviors of the graph of R on either side of the vertical asymptotes, completing Step 4).

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Conclusion	R negative	R is positive	R negative	R positive

So R is below the x-axis on $(-\infty, -2) \cup (0, 1)$ and R is above the x-axis on $(-2, 0) \cup (1, \infty)$.

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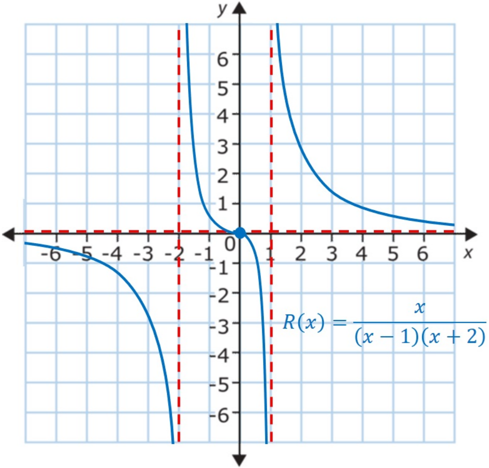
Solution (continued). In Step 7, we graph. Here, we reflect the asymptotes and the intercepts (but omit the points computed in Step 6):



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Page 211 Number 30. Follow Steps 1 through 7 to analyze the graph of $G(x) = \dfrac{x^2 - x - 12}{x+1}$.

Solution. For Step 1, we factor the numerator and denominator of G. Since $x^2 - x - 12 = (x+3)(x-4)$ then in lowest terms $\boxed{G(x) = \dfrac{(x+3)(x-4)}{x+1}}$. We see that the domain of G is all real numbers except $x = -1$; that is, the domain is $\boxed{(-\infty, -1) \cup (-1, \infty)}$.

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For Step 2, we write G in lowest terms. There are no common factors in the numerator and denominator, so nothing can be cancelled and G is in lowest terms as stated in Step 1.

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Solution (continued). In Step 3 we find intercepts. For x-intercepts, we set $y = G(x) = \dfrac{(x+3)(x-4)}{x+1} = 0$ and see that this implies that $x = -3$ or $x = 4$. So the $\boxed{x\text{-intercepts are } -3 \text{ and } 4}$. For the y-intercept, we set $x = 0$ to get $G(0) = \dfrac{((0)+3)((0)-4)}{(0)+1} = -12$. So the $\boxed{y\text{-intercept is } -12}$.

For Step 4, to find vertical asymptotes we set the denominator of G (in lowest terms) equal to 0 to get $x + 1 = 0$, or $x = -1$. So the $\boxed{\text{vertical asymptote is } x = -1}$.

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Page 211 Number 30 (continued 2)

Page 211 Number 30. Follow Steps 1 through 7 to analyze the graph of $G(x) = \dfrac{x^2 - x - 12}{x + 1}$.

Solution (continued). In Step 5, we find horizontal or oblique asymptotes. Notice that the numerator of G is of degree $n = 2$ and the denominator of G is degree $m = 1$. Since $n = 2 = m + 1$, then G has an oblique asymptote $y = ax + b$ which we find by long division (see Section 4.2). Consider:

$$
\begin{array}{r|rrrrrrr}
 & & & x & - & 2 \\
\cline{2-6}
x+1 & x^2 & - & x & - & 12 \\
 & x^2 & + & x & & \\
\cline{2-6}
 & & - & 2x & - & 12 \\
 & & - & 2x & - & 2 \\
\cline{3-6}
 & & & & - & 10
\end{array}
$$

So the $\boxed{\text{oblique asymptote is } y = x - 2}$.

Page 211 Number 30 (continued 3)

Solution (continued). In Step 6, we use the zeros of the numerator and denominator of G to divide the x-axis into intervals. The numerator is zero at $x = -3$ and $x = 4$ and the denominator is zero at $x = -1$, so we divide the x-axis into intervals: $(-\infty, -3)$, $(-3, -1)$, $(-1, 4)$, and $(4, \infty)$. Consider:

Interval	$(-\infty, -3)$	$(-3, -1)$	$(-1, 4)$	$(4, \infty)$
Test Value c	-4	-2	0	5
Value of $G(c)$	$(-1)(-8)/(-3)$	$(1)(-6)/(-1)$	$(3)(-4)/(1)$	$(8)(1)/(6)$
Conclusion	G negative	G is positive	G negative	G positive

So G is below the x-axis on $(-\infty, -3) \cup (-1, 4)$ and G is above the x-axis on $(-3, -1) \cup (4, \infty)$.

Notice that since G is positive on $(-3, -1)$ and G has a vertical asymptote at $x = -1$, then (in the notation of Page 193 Figure 29), $\lim_{x \to -1^-} G(x) = \infty$. Since G is negative on $(-1, 4)$ and G has a vertical asymptote at $x = -1$, then $\lim_{x \to -1^+} G(x) = -\infty$ (these observations give the behaviors of the graph of G on either side of the vertical asymptotes, completing Step 4).

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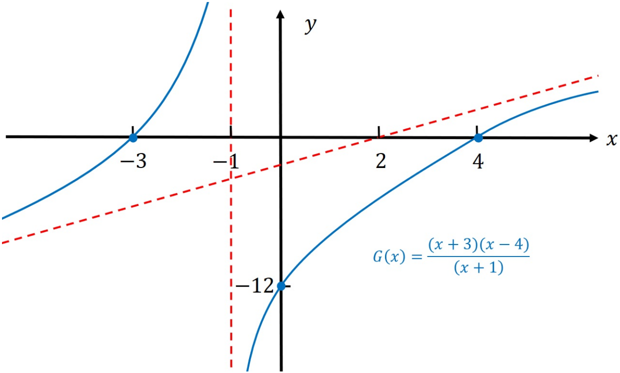
Solution (continued). In Step 7, we graph. Here, we reflect the asymptotes and the intercepts (but omit the points computed in Step 6):



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Page 211 Number 34

Page 211 Number 34. Follow Steps 1 through 7 to analyze the graph of $R(x) = \dfrac{x^2+3x-10}{x^2+8x+15}$.

Solution. For Step 1, we factor the numerator and denominator of R. Since $x^2+3x-10 = (x+5)(x-2)$ and $x^2+8x+15 = (x+5)(x+3)$ then $\boxed{R(x) = \dfrac{(x+5)(x-2)}{(x+5)(x+3)}}$. We see that the domain of R is all real numbers except $x=-5$ and $x=-3$; that is, the domain is $\boxed{(-\infty,-5)\cup(-5,-3)\cup(-3,\infty)}$.

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For Step 2, we write R in lowest terms, but carefully preserve the information about the domain. So we have $\boxed{R(x) = \dfrac{(x+5)(x-2)}{(x+5)(x+3)} = \dfrac{x-2}{x+3} \text{ for } x \neq -5}$.

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Solution. For Step 1, we factor the numerator and denominator of R. Since $x^2 + 3x - 10 = (x+5)(x-2)$ and $x^2 + 8x + 15 = (x+5)(x+3)$ then $\boxed{R(x) = \dfrac{(x+5)(x-2)}{(x+5)(x+3)}}$. We see that the domain of R is all real numbers except $x = -5$ and $x = -3$; that is, the domain is $\boxed{(-\infty, -5) \cup (-5, -3) \cup (-3, \infty)}$.

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$$\boxed{R(x) = \frac{(x+5)(x-2)}{(x+5)(x+3)} = \frac{x-2}{x+3} \text{ for } x \neq -5}.$$

Page 211 Number 34 (continued 1)

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Solution (continued). In Step 3 we find intercepts. We deal with the version of R in lowest terms. For x-intercepts, we set $y = R(x) = \dfrac{x-2}{x+3} = 0$ (for $x \neq -5$) and see that this implies that $x = 2$. So the $\boxed{x\text{-intercept is } 2}$. For the y-intercept, we set $x = 0$ to get $R(0) = \dfrac{(0) - 2}{(0) + 3} = -\dfrac{2}{3}$. So the $\boxed{y\text{-intercept is } -2/3}$.

For Step 4, to find vertical asymptotes we set the denominator of R (in lowest terms) equal to 0 to get $x + 3 = 0$, or $x = -3$. So the $\boxed{\text{vertical asymptote is } x = -3}$ (notice that when R is NOT in lowest terms, the denominator is also 0 at $x = -5$, but this does not yield a vertical asymptote of $x = -5$; see Theorem 2.2.A).

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Solution (continued). In Step 5, we find horizontal or oblique asymptotes. Notice that the numerator of R is of degree $n = 2$ and the denominator of R is degree $m = 1$. Since $n = m$, then R has a $\boxed{\text{horizontal asymptote of } y = a_n/b_m = (1)/(1) = 1}$.

In Step 6, we use the zeros of the numerator and denominator of R to divide the x-axis into intervals. The numerator is zero at $x = -5$ and $x = 2$ and the denominator is zero at $x = -5$ and $x = -3$, so we divide the x-axis into intervals: $(-\infty, -5)$, $(-5, -3)$, $(-3, 2)$, and $(2, \infty)$. Consider (we use the reduced version of R):

Interval	$(-\infty, -5)$	$(-5, -3)$	$(-3, 2)$	$(2, \infty)$
Test Value c	-6	-4	0	3
Value of $R(c)$	$(-8)/(-3)$	$(-6)/(-1)$	$(-2)/(3)$	$(1)/(6)$
Conclusion	R positive	R positive	R negative	R is positive

So R is $\boxed{\text{below the } x\text{-axis on } (-3, 2)}$ and R is $\boxed{\text{above the } x\text{-axis on } (-\infty, -5) \cup (-5, -3) \cup (2, \infty)}$.

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Solution (continued). Notice that since R is positive on $(-5,-3)$ and R has a vertical asymptote at $x=-3$, then (in the notation of Page 193 Figure 29), $\lim_{x\to -3^-} R(x) = \infty$. Since R is negative on $(-3,2)$ and R has a vertical asymptote at $x=-3$, then $\lim_{x\to -3^+} R(x) = -\infty$ (these observations give the behaviors of the graph of R on either side of the vertical asymptotes, completing Step 4).

Before we look at Step 7, notice that

$$R(x) = \frac{x^2+3x-10}{x^2+8x+15} = \frac{(x+5)(x-2)}{(x+5)(x+3)} = \frac{x-2}{x+3} \text{ if } x \neq -5.$$

Each of the properties above are satisfied by the function $g(x) = \dfrac{x-2}{x+3}$, except for the fact that the domain of g includes $x=-5$ (at which the value is $g(-5) = ((-5)-2)/((-5)+3) = 7/2$) and the domain of R does not. So the graph of R is the same as the graph of g, except that a "hole" is punched in the graph of R at $(-5, 7/2)$.

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Solution (continued). Notice that since R is positive on $(-5,-3)$ and R has a vertical asymptote at $x=-3$, then (in the notation of Page 193 Figure 29), $\lim_{x\to -3^-} R(x)=\infty$. Since R is negative on $(-3,2)$ and R has a vertical asymptote at $x=-3$, then $\lim_{x\to -3^+} R(x)=-\infty$ (these observations give the behaviors of the graph of R on either side of the vertical asymptotes, completing Step 4).

Before we look at Step 7, notice that

$$R(x)=\frac{x^2+3x-10}{x^2+8x+15}=\frac{(x+5)(x-2)}{(x+5)(x+3)}=\frac{x-2}{x+3} \text{ if } x\neq -5.$$

Each of the properties above are satisfied by the function $g(x)=\dfrac{x-2}{x+3}$, except for the fact that the domain of g includes $x=-5$ (at which the value is $g(-5)=((-5)-2)/((-5)+3)=7/2)$ and the domain of R does not. So the graph of R is the same as the graph of g, except that a "hole" is punched in the graph of R at $(-5,7/2)$.

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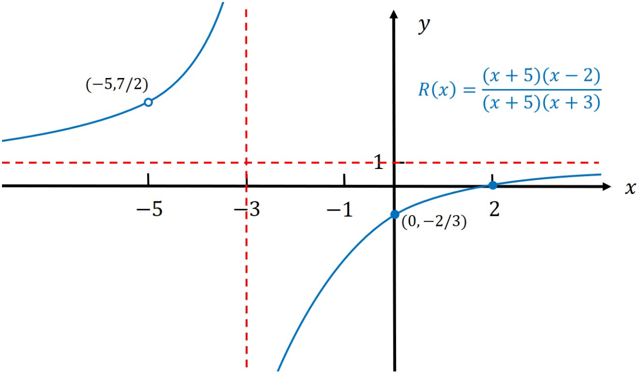

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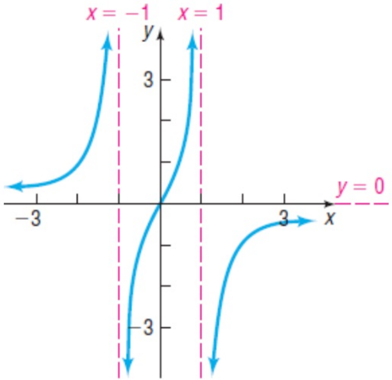
Page 212 Number 52. Find a rational function R that might have the given graph. (More than one answer might be possible.)



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Solution. Set $R(x) = p(x)/q(x)$ for polynomials p and q. Since the graph of R has vertical asymptotes at $x = -1$ and $x = 1$, then q needs to have factors of $x + 1$ and $x - 1$. Since the y-intercept is 0, then p must have a factor of $x - 0 = x$.

Since the graph of R has $y = 0$ as a horizontal asymptote, then the degree of the q must be greater than the degree of p, so we first try $p(x) = x$ and $q(x) = (x + 1)(x - 1) = x^2 - 1$ so that $R_1(x) = \dfrac{x}{x^2 - 1}$. Then R_1 has the correct domain, intercepts, and asymptotes.

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Page 212 Number 52 (continued)

Solution (continued). In Step 6, we use the zeros of the numerator and denominator of $R(x) = -\dfrac{x}{x^2 - 1}$ to divide the x-axis into intervals. The numerator is zero at $x = 0$ and the denominator is zero at $x = -1$ and $x = 1$, so we divide the x-axis into intervals: $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$. Consider:

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Test Value c	-2	$-1/2$	$1/2$	2
Value of $R(c)$	$-(-2)/(3)$	$-(-1/2)/(-3/4)$	$-(1/2)/(-3/4)$	$-(2)/(3)$
Conclusion	R positive	R is negative	R positive	R negative

So R is below the x-axis on $(-1, 0) \cup (1, \infty)$ and R is above the x-axis on $(-\infty, -1) \cup (0, 1)$.

Page 212 Number 52 (continued)

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Hence $\boxed{R(x) = -\dfrac{x}{x^2-1}}$ satisfies all the conditions we know from the given graph. □

Page 212 Number 52 (continued)

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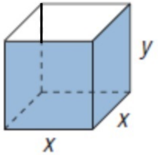
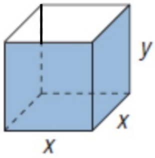
Page 213 Number 60

Page 213 Number 60. United Parcel Service has contracted you to design an open box with a square base that has a volume of 5000 cubic inches. See the illustration.



(a) Express the surface area S of the box as a function of x. **(b)** Using a graphing utility, graph the function found in part (a). **(c)** What is the minimum amount of cardboard that can be used to construct the box? **(d)** What are the dimensions of the box that minimize the surface area? **(e)** Why might UPS be interested in designing a box that minimizes the surface area?

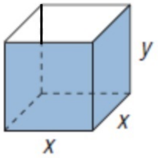
Page 213 Number 60 (continued 1)



Solution. **(a)** From the figure, we see that the box has length x, width x, and height y (we measure distances in inches). So the volume of the box is $V = (x)(x)(y) = x^2y$ and we are given that the volume is 5000 cubic inches, so $x^2y = 5000$. We can solve for y to get $y = 5000/x^2$. Now we find the total surface area by adding up the areas of the 5 sides of the box (it has no top) to get $S = x^2 + 4xy$, since the top and bottom have area x^2 and the sides have area xy, Substituting for y gives

$$S = x^2 + 4x\left(\frac{5000}{x^2}\right) = x^2 + \frac{20{,}000}{x} = \boxed{\frac{x^3 + 20{,}000}{x}}.$$

Page 213 Number 60 (continued 1)



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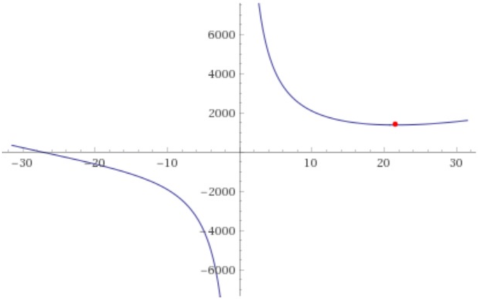
Page 213 Number 60 (continued 2)

Solution (continued). (b), **(c)**, **(d)** We do not have the techniques to algebraically answer these questions (you will have techniques once you take Calculus 1 [MATH 1910]). So we use Wolfram Alpha to graph and minimize:



The minimum (marked with the dot) occurs at some value slightly less than $x = 22$ inches.

Page 213 Number 60 (continued 3)

Solution (continued). In fact, Wolfram Alpha gives the exact value of x which minimizes function S and it is $x = 10\sqrt[3]{10} \approx 21.544$ inches. The minimum surface area is
$S(10\sqrt[3]{10}) = \dfrac{(10\sqrt[3]{10})^3 + 20{,}000}{10\sqrt[3]{10}} = 300 \times 10^{2/3} \approx 1392.477$ square inches. Since $y = 5000/x^2$, then when $x = 10\sqrt[3]{10}$ we have
$y = 5000/(10\sqrt[3]{10})^2 = 50 \times 10^{-2/3} \approx 10.772$ inches.

(e) UPS would be interested in designing a box that minimizes the surface area in order to minimize weight for a given volume. With less weight, they can save money on the material from which the box is made and on the energy it takes to move the box from factory to delivery. Their goal certainly includes minimizing cost (in order to maximize profit).
…now why they want an open top box… □

Page 213 Number 60 (continued 3)

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