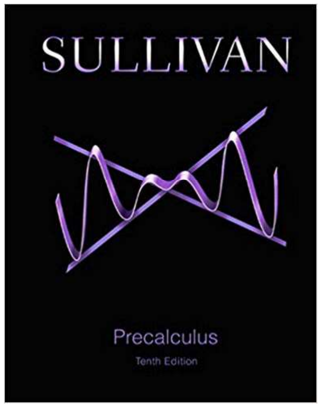
Precalculus 1 (Algebra)

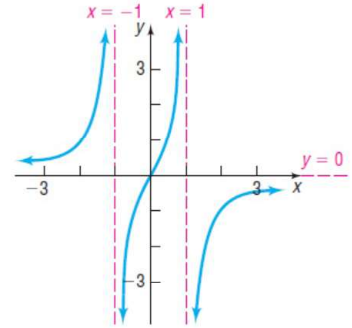
Chapter 4. Polynomial and Rational Functions

4.4. Polynomial and Rational Inequalities—Exercises, Examples, Proofs



Page 218 Number 7

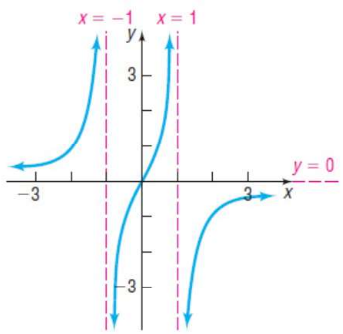
Page 218 Number 7. Use the graph of the function f to solve the inequalities **(a)** $f(x) > 0$ and **(b)** $f(x) \leq 0$.



Solution. (a) We have $f(x) > 0$ when the graph of $y = f(x)$ is strictly above the x-axis (that is, when the y-coordinate of points is greater than 0). So $f(x) > 0$ for $\boxed{x \text{ in } (-\infty, -1) \cup (0, 1)}$. □

Page 218 Number 7 (continued)

Page 218 Number 7. Use the graph of the function f to solve the inequalities **(a)** $f(x) > 0$ and **(b)** $f(x) \leq 0$.



Solution (continued). (b) We have $f(x) \leq 0$ when the graph of $y = f(x)$ is below or on the x-axis (that is, when the y-coordinate of points is less than or equal to 0). So $f(x) \leq 0$ for $\boxed{x \text{ in } (-1, 0] \cup (1, \infty)}$. □

Page 219 Number 26

Page 219 Number 26. Solve the inequality $(x+1)(x+2)(x+3) \leq 0$ algebraically.

Solution. We follow the 4 step method just introduced. Step 1 is given in the statement of the question, where we take $f(x) = (x+1)(x+2)(x+3)$. For Step 2, we see that the left side of the inequality, $f(x)$, is already factored and the real zeros of $f(x) = (x+1)(x+2)(x+3)$ are $x = -3$, $x = -2$, and $x = -1$. For Step 3, we divide the real number line $\mathbb{R} = (-\infty, \infty)$ into intervals by removing the zeros of f to get: $(-\infty, -3)$, $(-3, -2)$, $(-2, -1)$, $(-1, \infty)$. For Step 4, consider

Interval	$(-\infty, -3)$	$(-3, -2)$	$(-2, -1)$	$(-1, \infty)$
Test Value c	-4	$-5/2$	$-3/2$	0
Value of $f(c)$	$(-3)(-2)(-1)$	$(-3/2)(-1/2)(1/2)$	$(-1/2)(1/2)(3/2)$	$(1)(2)(3)$
Conclusion	f negative	f positive	f negative	f positive

So $f(x) \leq 0$ for $\boxed{x \text{ in } (-\infty, -3] \cup [-2, -1]}$. □

Page 219 Number 48

Page 219 Number 48. Solve the inequality $\dfrac{(2-x)^3(3x-2)}{x^3+1} < 0$ algebraically.

Solution. We follow the 4 step method just introduced. Step 1 is given in the statement of the question, where we take $f(x) = \dfrac{(2-x)^3(3x-2)}{x^3+1}$. For Step 2, we see that the left side of the inequality, $f(x)$, is already factored and the real zeros of f are $x = 2$ and $x = 2/3$, and the only zero of the denominator is $x = \sqrt[3]{-1} = -1$. For Step 3, we divide the real number line $\mathbb{R} = (-\infty, \infty)$ into intervals by removing these zeros to get: $(-\infty, -1)$, $(-1, 2/3)$, $(2/3, 2)$, $(2, \infty)$. For Step 4, consider

Interval	$(-\infty,-1)$	$(-1,2/3)$	$(2/3,2)$	$(2,\infty)$
Test Value c	-2	0	1	3
Value of $f(c)$	$(4)^3(-8)/(-7)$	$(2)^3(-2)/(1)$	$(1)^3(1)/(2)$	$(-1)^3(7)/(28)$
Conclusion	f positive	f negative	f positive	f negative

So $f(x) < 0$ for $\boxed{x \text{ in } (-1, 2/3) \cup (2, \infty)}$. □

Page 219 Number 64

Page 219 Number 64. Consider $f(x) = \dfrac{x^3 - 6x^2 + 9x - 4}{x^2 + x - 20}$. **(a)** Graph f, and **(b)** solve $f(x) \geq 0$. HINT: $x^3 - 6x^2 + 9x - 4 = (x-4)(x-1)^2$

Solution. (a) We follow the 7 steps for graphing a rational function introduced in Section 4.3. For Step 1, we factor the numerator and denominator of f. The numerator is factored using the hint and the denominator easily factors as $x^2 + x - 20 = (x-4)(x+5)$. So, in fact, $f(x) = \dfrac{(x-4)(x-1)^2}{(x-4)(x+5)}$. We see that the domain of f is all real numbers except $x = -5$ and $x = 4$; that is, the domain is $(-\infty, -5) \cup (-5, 4) \cup (4, \infty)$. For Step 2, we write f in lowest terms, but carefully preserve the information about the domain. So we have $f(x) = \dfrac{(x-4)(x-1)^2}{(x-4)(x+5)} = \dfrac{(x-1)^2}{x+5}$ for $x \neq 4$.

Page 219 Number 64 (continued 1)

Solution (continued). For Step 3, we find the x-intercepts by solving $f(x) = 0$ for x in the domain of f. So we set the numerator of f equal to 0, $(x-4)(x-1)^2 = 0$, and we see that this implies that $x = 1$ and $x = 4$; however, 4 is not in the domain of f so the only x-intercept is $x = 1$. Notice that $x = 1$ is a zero of f of multiplicity 2, so the graph touches the $x - axis$ at $x = 1$. For the y-intercept, we set $x = 0$ to get $f(0) = \dfrac{(-4)(-1)^2}{(-4)(5)} = \dfrac{1}{5}$. For Step 4, to find vertical asymptotes we set the denominator of f (in lowest terms) equal to 0 to get $x + 5 = 0$ or $x = -5$. So there is a vertical asymptote of $x = -5$ (notice that when f is NOT in lowest terms, the denominator is also 0 at $x = 4$, but this does not yield a vertical asymptote at $x = 4$; see Theorem 2.2.A). In Step 5, we find horizontal or oblique asymptotes. Notice that the denominator of f is degree $m = 2$ and the numerator is of degree $n = 3 = m + 1$, so f has an oblique asymptote $y = ax + b$ which we find by long division (see Section 4.2); we can deal with the version of f in lowest terms:

Page 219 Number 64 (continued 2)

Solution (continued).

$$
\begin{array}{r|rrrrr}
 & & & x & - & 7 \\ \hline
x+5 & x^2 & - & 2x & + & 1 \\
 & x^2 & + & 5x & & \\ \hline
 & & - & 7x & + & 1 \\
 & & - & 7x & - & 35 \\ \hline
 & & & & & 36
\end{array}
$$

So the oblique asymptote is $y = x - 7$. For Step 6, we have that f is 0 at $x = 1$ and is not defined at $x = -5$ and $x = 4$, so we divide the x-axis into intervals: $(-\infty, -5)$, $(-5, 1)$, $(1, 4)$, $(4, \infty)$. Consider (we use the reduced version of f):

Interval	$(-\infty,-5)$	$(-5,1)$	$(1,4)$	$(4,\infty)$
Test Value c	-6	0	2	5
Value of $f(c)$	$(-7)^2/(-1)$	$(-1)^2/(5)$	$(1)^2/(7)$	$(4)^2/(10)$
Conclusion	f negative	f positive	f positive	f positive

So f is below the x-axis on $(-\infty, -5)$ and f is above the x-axis on $(-5, 1) \cup (1, 4) \cup (4, \infty)$.

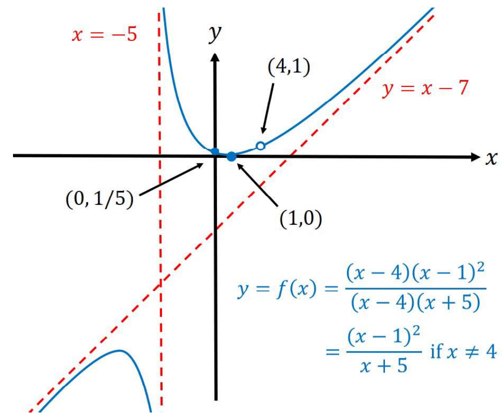
Page 219 Number 64 (continued 3)

Solution (continued). Notice that since f is negative on $(-\infty, -5)$ and f has a vertical asymptote at $x = -5$, then (in the notation of Page 193 Figure 29), $\lim_{x\to -5^-} f(x) = -\infty$. Since f is positive on $(-5, 1)$ and f has a vertical asymptote at $x = -5$, then $\lim_{x\to -5^+} f(x) = \infty$ (this gives the behavior of the graph of f on either side of the vertical asymptote, completing Step 4).

Before graphing, we notice that $f(x) = \dfrac{(x-4)(x-1)^2}{(x-4)(x+5)} = \dfrac{(x-1)^2}{x+5}$ for $x \neq 4$. Each of the properties above are satisfied by the function $g(x) = \frac{(x-1)^2}{x+5}$, except for the fact that the domain of g includes $x = 4$ (at which the value is $g(4) = ((4)-1)^2/((4)+5) = 9/9 = 1$) and the domain of f does not. The effect of this is that the graph of f and g are the same, except at the point $(4, 1)$ where g includes this point and f does not. So the graph of f is the same as the graph of g, except that a "hole" is punched in the graph of f at $(4, 1)$. All this information combines to give the following graph.

Page 219 Number 64 (continued 4)

Solution (continued).



(b) We see from the graph that $f(x) \geq 0$ for x in $\boxed{(-5, 4) \cup (4, \infty)}$. □

Page 220 Number 76

Page 220 Number 76. Suppose that the government imposes a \$1000-per-day tax on the bicycle manufacturer so that the daily cost C of manufacturing x bicycles is now given by $C(x) = 80x + 6000$. The average daily cost $\overline{C}$ is given by $\overline{C}(x) = (80x + 6000)/x$. How many bicycles must be produced each day for the average cost to be no more than \$100?

Solution. For the average cost $\overline{C}$ to be no more than \$100 requires $\overline{C}(x) = (80x + 6000)/x \leq 100$, so we solve this inequality algebraically by following the 4 steps. For Step 1, define $f(x) = \overline{C}(x) - 100 = (80x + 6000)/x - 100 = (6000 - 20x)/x$ so that the given inequality becomes $f(x) = (6000 - 20x)/x \leq 0$. For Step 2, we see that the left side of the inequality, $f(x) = (6000 - 20x)/x$, is already factored and the real zero of f is $x = 6000/20 = 300$, and the only zero of the denominator is $x = 0$. However, notice that x is a number of bicycles so we must have $x \geq 0$. But we also just saw that $x \neq 0$; this means the domain of f is $(0, \infty)$.

Page 220 Number 76 (continued)

Page 220 Number 76. Suppose that the government imposes a \$1000-per-day tax on the bicycle manufacturer so that the daily cost C of manufacturing x bicycles is now given by $C(x) = 80x + 6000$. The average daily cost $\overline{C}$ is given by $\overline{C}(x) = (80x + 6000)/x$. How many bicycles must be produced each day for the average cost to be no more than \$100?

Solution (continued). For Step 3, we divide the domain of f, $(0, \infty)$, into intervals by removing the points $x = 0$ and $x = 300$ to get: $(0, 300)$, $(300, \infty)$. For Step 4, consider

Interval	$(0, 300)$	$(300, \infty)$
Test Value c	50	301
Value of $f(c)$	$(5000)/(50)$	$(-20)/(301)$
Conclusion	f positive	f negative

So $f(x) \leq 0$ (and $\overline{C}(x) \leq 100$) for x in $[300, \infty)$. So the bicycle manufacturer should $\boxed{\text{produce at least 300 bicycles each day}}$. □