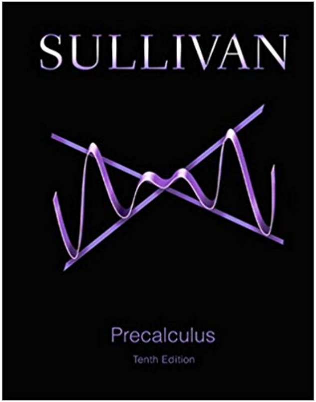
# Precalculus 1 (Algebra)

**Chapter 4. Polynomial and Rational Functions**

4.4. Polynomial and Rational Inequalities—Exercises, Examples, Proofs



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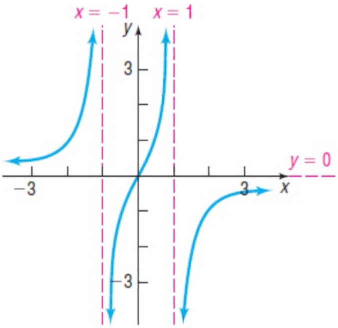
# Page 218 Number 7

**Page 218 Number 7.** Use the graph of the function $f$ to solve the inequalities **(a)** $f(x) > 0$ and **(b)** $f(x) \leq 0$.




**Solution. (a)** We have $f(x) > 0$ when the graph of $y = f(x)$ is strictly above the $x$-axis (that is, when the $y$-coordinate of points is greater than 0). So $f(x) > 0$ for $\boxed{x \text{ in } (-\infty, -1) \cup (0, 1)}$. □

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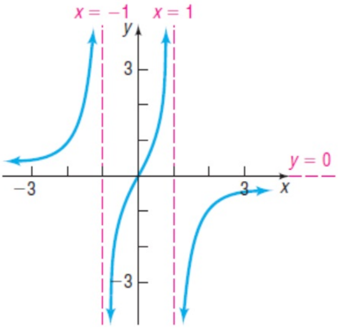


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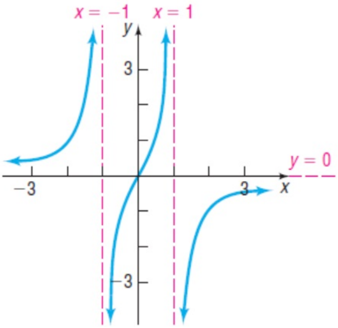


**Solution (continued).** **(b)** We have $f(x) \leq 0$ when the graph of $y = f(x)$ is below or on the $x$-axis (that is, when the $y$-coordinate of points is less than or equal to 0). So $f(x) \leq 0$ for $\boxed{x \text{ in } (-1, 0] \cup (1, \infty)}$. □

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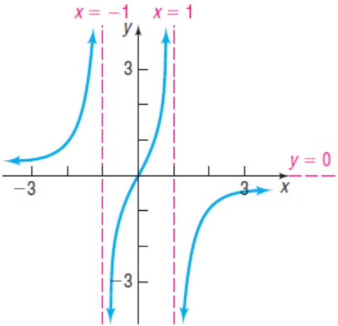


**Solution (continued).** **(b)** We have $f(x) \leq 0$ when the graph of $y = f(x)$ is below or on the $x$-axis (that is, when the $y$-coordinate of points is less than or equal to 0). So $f(x) \leq 0$ for $\boxed{x \text{ in } (-1, 0] \cup (1, \infty)}$. □

# Page 219 Number 26

**Page 219 Number 26.** Solve the inequality $(x+1)(x+2)(x+3) \leq 0$ algebraically.

**Solution.** We follow the 4 step method just introduced. Step 1 is given in the statement of the question, where we take $f(x) = (x+1)(x+2)(x+3)$.

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| **Interval** | $(-\infty, -3)$ | $(-3, -2)$ | $(-2, -1)$ | $(-1, \infty)$ |
|---|---|---|---|---|
| **Test Value** $c$ | $-4$ | $-5/2$ | $-3/2$ | $0$ |
| **Value of** $f(c)$ | $(-3)(-2)(-1)$ | $(-3/2)(-1/2)(1/2)$ | $(-1/2)(1/2)(3/2)$ | $(1)(2)(3)$ |
| **Conclusion** | $f$ negative | $f$ positive | $f$ negative | $f$ positive |

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| **Value of** $f(c)$ | $(4)^3(-8)/(-7)$ | $(2)^3(-2)/(1)$ | $(1)^3(1)/(2)$ | $(-1)^3(7)/(28)$ |
| **Conclusion** | $f$ positive | $f$ negative | $f$ positive | $f$ negative |

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So $f(x) < 0$ for $x$ in $(-1, 2/3) \cup (2, \infty)$. □

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# Page 219 Number 64

**Page 219 Number 64.** Consider $f(x) = \dfrac{x^3 - 6x^2 + 9x - 4}{x^2 + x - 20}$.
**(a)** Graph $f$, and **(b)** solve $f(x) \geq 0$. HINT:
$x^3 - 6x^2 + 9x - 4 = (x - 4)(x - 1)^2$

**Solution. (a)** We follow the 7 steps for graphing a rational function introduced in Section 4.3. For Step 1, we factor the numerator and denominator of $f$. The numerator is factored using the hint and the denominator easily factors as $x^2 + x - 20 = (x - 4)(x + 5)$. So, in fact, $f(x) = \dfrac{(x-4)(x-1)^2}{(x-4)(x+5)}$. We see that the domain of $f$ is all real numbers except $x = -5$ and $x = 4$; that is, the domain is $(-\infty, -5) \cup (-5, 4) \cup (4, \infty)$.

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$$f(x) = \frac{(x-4)(x-1)^2}{(x-4)(x+5)} = \frac{(x-1)^2}{x+5} \text{ for } x \neq 4.$$

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# Page 219 Number 64 (continued 1)

**Solution (continued).** For Step 3, we find the $x$-intercepts by solving $f(x) = 0$ for $x$ in the domain of $f$. So we set the numerator of $f$ equal to 0, $(x-4)(x-1)^2 = 0$, and we see that this implies that $x = 1$ and $x = 4$; however, 4 is not in the domain of $f$ so the only $x$-intercept is $x = 1$. Notice that $x = 1$ is a zero of $f$ of multiplicity 2, so the graph touches the $x - axis$ at $x = 1$. For the $y$-intercept, we set $x = 0$ to get $f(0) = \dfrac{(-4)(-1)^2}{(-4)(5)} = \dfrac{1}{5}$. For Step 4, to find vertical asymptotes we set the denominator of $f$ (in lowest terms) equal to 0 to get $x + 5 = 0$ or $x = -5$. So there is a vertical asymptote of $x = -5$ (notice that when $f$ is NOT in lowest terms, the denominator is also 0 at $x = 4$, but this does not yield a vertical asymptote at $x = 4$; see Theorem 2.2.A).

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**Solution (continued).** For Step 3, we find the $x$-intercepts by solving $f(x) = 0$ for $x$ in the domain of $f$. So we set the numerator of $f$ equal to 0, $(x-4)(x-1)^2 = 0$, and we see that this implies that $x = 1$ and $x = 4$; however, 4 is not in the domain of $f$ so the only $x$-intercept is $x = 1$. Notice that $x = 1$ is a zero of $f$ of multiplicity 2, so the graph touches the $x - axis$ at $x = 1$. For the $y$-intercept, we set $x = 0$ to get $f(0) = \dfrac{(-4)(-1)^2}{(-4)(5)} = \dfrac{1}{5}$. For Step 4, to find vertical asymptotes we set the denominator of $f$ (in lowest terms) equal to 0 to get $x + 5 = 0$ or $x = -5$. So there is a vertical asymptote of $x = -5$ (notice that when $f$ is NOT in lowest terms, the denominator is also 0 at $x = 4$, but this does not yield a vertical asymptote at $x = 4$; see Theorem 2.2.A). In Step 5, we find horizontal or oblique asymptotes. Notice that the denominator of $f$ is degree $m = 2$ and the numerator is of degree $n = 3 = m + 1$, so $f$ has an oblique asymptote $y = ax + b$ which we find by long division (see Section 4.2); we can deal with the version of $f$ in lowest terms:

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**Solution (continued).** For Step 3, we find the $x$-intercepts by solving $f(x) = 0$ for $x$ in the domain of $f$. So we set the numerator of $f$ equal to 0, $(x-4)(x-1)^2 = 0$, and we see that this implies that $x = 1$ and $x = 4$; however, 4 is not in the domain of $f$ so the only $x$-intercept is $x = 1$. Notice that $x = 1$ is a zero of $f$ of multiplicity 2, so the graph touches the $x - axis$ at $x = 1$. For the $y$-intercept, we set $x = 0$ to get $f(0) = \dfrac{(-4)(-1)^2}{(-4)(5)} = \dfrac{1}{5}$. For Step 4, to find vertical asymptotes we set the denominator of $f$ (in lowest terms) equal to 0 to get $x + 5 = 0$ or $x = -5$. So there is a vertical asymptote of $x = -5$ (notice that when $f$ is NOT in lowest terms, the denominator is also 0 at $x = 4$, but this does not yield a vertical asymptote at $x = 4$; see Theorem 2.2.A). In Step 5, we find horizontal or oblique asymptotes. Notice that the denominator of $f$ is degree $m = 2$ and the numerator is of degree $n = 3 = m + 1$, so $f$ has an oblique asymptote $y = ax + b$ which we find by long division (see Section 4.2); we can deal with the version of $f$ in lowest terms:

# Page 219 Number 64 (continued 2)

**Solution (continued).**

$$\begin{array}{r|rrrrrrr} & & & x & - & 7 \\ \hline x+5 & x^2 & - & 2x & + & 1 \\ & x^2 & + & 5x & & \\ \hline & & - & 7x & + & 1 \\ & & - & 7x & - & 35 \\ \hline & & & & & 36 \end{array}$$

So the oblique asymptote is $y = x - 7$. For Step 6, we have that $f$ is 0 at $x = 1$ and is not defined at $x = -5$ and $x = 4$, so we divide the $x$-axis into intervals: $(-\infty, -5)$, $(-5, 1)$, $(1, 4)$, $(4, \infty)$. Consider (we use the reduced version of $f$):

| | | | | |
|---|---|---|---|---|
| **Interval** | $(-\infty, -5)$ | $(-5, 1)$ | $(1, 4)$ | $(4, \infty)$ |
| **Test Value** $c$ | $-6$ | $0$ | $2$ | $5$ |
| **Value of** $f(c)$ | $(-7)^2/(-1)$ | $(-1)^2/(5)$ | $(1)^2/(7)$ | $(4)^2/(10)$ |
| **Conclusion** | $f$ negative | $f$ positive | $f$ positive | $f$ positive |

# Page 219 Number 64 (continued 2)

**Solution (continued).**

$$\begin{array}{r|l} & x - 7 \\ \hline x+5 & x^2 - 2x + 1 \\ & x^2 + 5x \\ \hline & \quad - 7x + 1 \\ & \quad - 7x - 35 \\ \hline & \qquad\qquad 36 \end{array}$$

So the oblique asymptote is $y = x - 7$. For Step 6, we have that $f$ is 0 at $x = 1$ and is not defined at $x = -5$ and $x = 4$, so we divide the $x$-axis into intervals: $(-\infty, -5)$, $(-5, 1)$, $(1, 4)$, $(4, \infty)$. Consider (we use the reduced version of $f$):

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| **Conclusion** | $f$ negative | $f$ positive | $f$ positive | $f$ positive |

So $f$ is below the $x$-axis on $(-\infty, -5)$ and $f$ is above the $x$-axis on $(-5, 1) \cup (1, 4) \cup (4, \infty)$.

# Page 219 Number 64 (continued 2)

**Solution (continued).**

$$\begin{array}{r|rrrrr} & & x & - & 7 \\ \hline x+5 & x^2 & - & 2x & + & 1 \\ & x^2 & + & 5x & & \\ \hline & & - & 7x & + & 1 \\ & & - & 7x & - & 35 \\ \hline & & & & & 36 \end{array}$$

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So $f$ is below the $x$-axis on $(-\infty, -5)$ and $f$ is above the $x$-axis on $(-5, 1) \cup (1, 4) \cup (4, \infty)$.

# Page 219 Number 64 (continued 3)

**Solution (continued).** Notice that since $f$ is negative on $(-\infty, -5)$ and $f$ has a vertical asymptote at $x = -5$, then (in the notation of Page 193 Figure 29), $\lim_{x\to -5^-} f(x) = -\infty$. Since $f$ is positive on $(-5, 1)$ and $f$ has a vertical asymptote at $x = -5$, then $\lim_{x\to -5^+} f(x) = \infty$ (this gives the behavior of the graph of $f$ on either side of the vertical asymptote, completing Step 4).

Before graphing, we notice that $f(x) = \dfrac{(x-4)(x-1)^2}{(x-4)(x+5)} = \dfrac{(x-1)^2}{x+5}$ for $x \neq 4$. Each of the properties above are satisfied by the function $g(x) = \frac{(x-1)^2}{x+5}$, except for the fact that the domain of $g$ includes $x = 4$ (at which the value is $g(4) = ((4)-1)^2/((4)+5) = 9/9 = 1$) and the domain of $f$ does not. The effect of this is that the graph of $f$ and $g$ are the same, except at the point $(4, 1)$ where $g$ includes this point and $f$ does not. So the graph of $f$ is the same as the graph of $g$, except that a "hole" is punched in the graph of $f$ at $(4, 1)$. All this information combines to give the following graph.

# Page 219 Number 64 (continued 3)

**Solution (continued).** Notice that since $f$ is negative on $(-\infty, -5)$ and $f$ has a vertical asymptote at $x = -5$, then (in the notation of Page 193 Figure 29), $\lim_{x \to -5^-} f(x) = -\infty$. Since $f$ is positive on $(-5, 1)$ and $f$ has a vertical asymptote at $x = -5$, then $\lim_{x \to -5^+} f(x) = \infty$ (this gives the behavior of the graph of $f$ on either side of the vertical asymptote, completing Step 4).

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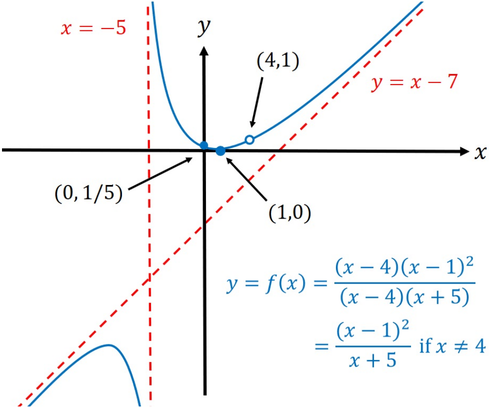
# Page 219 Number 64 (continued 4)

**Solution (continued).**




**(b)** We see from the graph that $f(x) \geq 0$ for $x$ in $\boxed{(-5, 4) \cup (4, \infty)}$. □

# Page 220 Number 76

**Page 220 Number 76.** Suppose that the government imposes a \$1000-per-day tax on the bicycle manufacturer so that the daily cost $C$ of manufacturing $x$ bicycles is now given by $C(x) = 80x + 6000$. The average daily cost $\overline{C}$ is given by $\overline{C}(x) = (80x + 6000)/x$. How many bicycles must be produced each day for the average cost to be no more than \$100?

**Solution.** For the average cost $\overline{C}$ to be no more than \$100 requires $\overline{C}(x) = (80x + 6000)/x \leq 100$, so we solve this inequality algebraically by following the 4 steps. For Step 1, define $f(x) = \overline{C}(x) - 100 = (80x + 6000)/x - 100 = (6000 - 20x)/x$ so that the given inequality becomes $f(x) = (6000 - 20x)/x \leq 0$.

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**Solution (continued).** For Step 3, we divide the domain of $f$, $(0, \infty)$, into intervals by removing the points $x = 0$ and $x = 300$ to get: $(0, 300)$, $(300, \infty)$. For Step 4, consider

| | | |
|---|---|---|
| **Interval** | $(0, 300)$ | $(300, \infty)$ |
| **Test Value** $c$ | 50 | 301 |
| **Value of** $f(c)$ | $(5000)/(50)$ | $(-20)/(301)$ |
| **Conclusion** | $f$ positive | $f$ negative |

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So $f(x) \leq 0$ (and $\overline{C}(x) \leq 100$) for $x$ in $[300, \infty)$. So the bicycle manufacturer should produce at least 300 bicycles each day. □

# Page 220 Number 76 (continued)

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