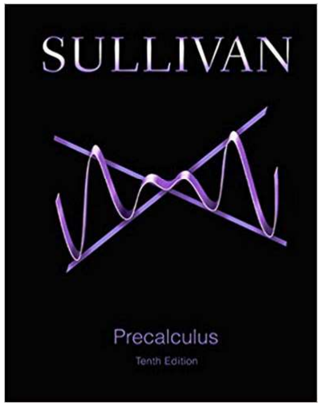
# Precalculus 1 (Algebra)

**Chapter 4. Polynomial and Rational Functions**

4.5. The Real Zeros of a Polynomial Function—Exercises, Examples, Proofs








## Cubic Formula Problem 1

**Cubic Formula Problem 1.** Show that the general cubic equation $y^3 + by^2 + cy + d = 0$ can be transformed into an equation of the form $x^3 + px + q = 0$ by using the substitution $y = x - b/3$.

**Solution.** With $y = x - b/3$, we have

$$y^2 = (x - b/3)^2 = x^2 - 2bx/3 + b^2/9 \text{ and}$$

$$y^3 = (x - b/3)(x^2 - 2bx/3 + b^2/9) = x^3 - 2bx^2/3 + b^2x/9 - bx^2/3$$
$$+2b^2x/9 - b^3/27 = x^3 - 3bx^2/3 + 3b^2x/9 - b^3/27$$
$$= x^3 - bx^2 + b^2x/3 - b^3/27.$$

We then have

$$0 = y^3 + by^2 + cy + d = (x^3 - bx^2 + b^2x/3 - b^3/27) + b(x^2 - 2bx/3 + b^2/9)$$

$$+c(x - b/3) + d = x^3 + (-b^2/3 + c)x + (2b^3/27 - bc/3 + d).$$





## Cubic Formula Problem 1 (continued)

**Cubic Formula Problem 1.** Show that the general cubic equation $y^3 + by^2 + cy + d = 0$ can be transformed into an equation of the form $x^3 + px + q = 0$ by using the substitution $y = x - b/3$.

**Solution.** ...

$$0 = y^3 + by^2 + cy + d = x^3 + (-b^2/3 + c)x + (2b^3/27 - bc/3 + d).$$

We then have $x^3 + px + q = 0$ where $p = c - b^2/3$ and $q = 2b^3/27 - bc/3 + d$. □





## Cubic Formula Problem 2

**Cubic Formula Problem 2.** In the equation $x^3 + px + q = 0$, replace $x$ by $H + K$. Let $3HK = -p$, and show that $H^3 + K^3 = -q$.

**Solution.** With $x = H + K$ we have

$$x^3 = (H + K)^3 = (H + K)(H^2 + 2HK + K^2) = H^3 + 2H^2K + HK^2 + H^2K$$

$$+2HK^2 + K^3 = H^3 + 3H^2K + 3HK^2 + K^3.$$

With $3HK = -p$, or $p = -3HK$ we then have

$$0 = x^3 + px + q = (H^3 + 3H^2K + 3HK^2 + K^3) + (-3HK)(H + K) + q$$

$$= H^3 + 3H^2K + 3HK^2 + K^3 - 3H^2K - 3HK^2 + q = H^3 + K^3 + q$$

or $H^3 + K^3 = -q$, as claimed. □

## Cubic Formula Problem 3

**Cubic Formula Problem 3.** Based on Cubic Formula Problem 2, we have two equations $3HK = -p$ and $H^3 + K^3 = -q$. Solve for $K$ in $3HK = -p$ and substitute into $H^3 + K^3 = -q$. Then show that $H = \sqrt[3]{\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$. HINT: Look for an equation that is quadratic in form.

**Solution.** Since $3HK = -p$, then $K = -p/(3H)$. So $H^3 + K^3 = -q$ implies that $-q = H^3 + (-p/(3H))^3 = H^3 - p^3/(27H^3)$ or, multiplying both sides of this last equation by $H^3$, $-qH^3 = H^6 - p^3/27$ or $H^6 + qH^3 - p^3/27 = 0$ or $(H^3)^2 + q(H^3) - p^3/27 = 0$. So we have a quadratic equation in the unknown $H^3$ and so we can solve for $H^3$ using the quadratic formula to get:

$$H^3 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(q) \pm \sqrt{(q)^2 - 4(1)(-p^3/27)}}{2(1)} = \ldots$$





## Cubic Formula Problem 3 (continued)

**Solution (continued).** ...

$$H^3 = \frac{-q \pm \sqrt{q^2 + 4p^3/27}}{2} = \frac{-q}{2} \pm \frac{\sqrt{q^2 + 4p^3/27}}{2}$$

$$= \frac{-q}{2} \pm \frac{\sqrt{q^2 + 4p^3/27}}{\sqrt{4}} = \frac{-q}{2} \pm \sqrt{\frac{q^2 + 4p^3/27}{4}}$$

$$= \frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

Hence,

$$H = \sqrt[3]{\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

□





## Cubic Formula Problem 4

**Cubic Formula Problem 4.** Use the solution for $H$ from Cubic Formula Problem 3 and the equation $H^3 + K^3 = -q$ to show that $K = \sqrt[3]{\frac{-q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$.

**Solution.** With $H = \sqrt[3]{\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$, we have

$$-q = H^3 + K^3 = \frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} + K^3 \text{ or}$$

$$K^3 = -q - \left(\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right) = \frac{-q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

So

$$K = \sqrt[3]{\frac{-q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

□





## Cubic Formula Problem 5

**Cubic Formula Problem 5.** Use the results from Cubic Formula Problems 2 to 4 to show that the solution of $x^3 + px + q = 0$ is

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

**Solution.** First, we have $x = H + K$, so

$$x = \sqrt[3]{\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

$$= \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

Now with $x = H + K$, $3HK = -p$ (or $p = -3HK$), and $H^3 + K^3 = -q$ (or $q = -H^3 - K^3$) from Cubic Formula Problem 2, we have ...

## Cubic Formula Problem 5 (continued)

**Cubic Formula Problem 5.** Use the results from Cubic Formula Problems 2 to 4 to show that the solution of $x^3 + px + q = 0$ is

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

**Solution (continued).** . . .

$$x^3 + px + q = (H+K)^3 + (-3HK)(H+K) + (-H^3 - K^3)$$
$$= (H+K)(H^2 + 2HK + K^2) - 3H^2K - 3HK^2 - H^3 - K^3$$
$$= (H^3 + 2H^2K + HK^2 + H^2K + 2HK^2 + K^3) - 3H^2K - 3HK^2 - H^3 - K^3$$
$$= (H^3 + 3H^2K + 3HK^2 + K^3) - 3H^2K - 3HK^2 - H^3 - K^3 = 0,$$

as claimed. □





## Cubic Formula Problem 6

**Cubic Formula Problem 6.** Use the result of Cubic Formula Problem 5 to solve the equation $x^3 - 6x - 9 = 0$.

**Solution.** Here, we have $p = -6$ and $q = -9$. Since by Cubic Problem 5,

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

we have (simplifying a little up front and skipping steps)

$$x = \sqrt[3]{\frac{9}{2} + \sqrt{\frac{81}{4} + \frac{-216}{27}}} + \sqrt[3]{\frac{9}{2} - \sqrt{\frac{81}{4} + \frac{-216}{27}}}$$
$$= \sqrt[3]{\frac{9}{2} + \sqrt{\frac{49}{4}}} + \sqrt[3]{\frac{9}{2} - \sqrt{\frac{49}{4}}} = \sqrt[3]{\frac{9}{2} + \frac{7}{2}} + \sqrt[3]{\frac{9}{2} - \frac{7}{2}}$$
$$= \sqrt[3]{8} + \sqrt[3]{1} = 2 + 1 = \boxed{3}.$$ □





## Cubic Formula Problem 7

**Cubic Formula Problem 7.** Use the result of Cubic Formula Problem 5 to solve $x^3 + 3x - 14 = 0$. Use a calculator to give a decimal approximation of the solution to the equation.

**Solution.** Here, we have $p = 3$ and $q = -14$. Since by Cubic Formula Problem 5,

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

we have (simplifying a little up front and skipping steps)

$$x = \sqrt[3]{\frac{14}{2} + \sqrt{\frac{196}{4} + \frac{27}{27}}} + \sqrt[3]{\frac{14}{2} - \sqrt{\frac{196}{4} + \frac{27}{27}}}$$
$$= \sqrt[3]{7 + \sqrt{49+1}} + \sqrt[3]{7 - \sqrt{49+1}} = \sqrt[3]{7 + \sqrt{50}} + \sqrt[3]{7 - \sqrt{50}} \ldots$$





## Cubic Formula Problem 7 (continued)

**Cubic Formula Problem 7.** Use the result of Cubic Formula Problem 5 to solve $x^3 + 3x - 14 = 0$. Use a calculator to give a decimal approximation of the solution to the equation.

**Solution (continued).** . . .

$$x = \boxed{\sqrt[3]{7 + 5\sqrt{2}} + \sqrt[3]{7 - 5\sqrt{2}}},$$

since $\sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$. If we plug $x$ into a calculator, we find that it simplifies to $\boxed{x = 2}$. □

## Cubic Formula Problem 8

**Cubic Formula Problem 8.** Use the methods of this section to solve the equation $x^3 + 3x - 14 = 0$.

**Solution.** We find the factors of the leading coefficient $a_3 = 1$ and the constant term $a_0 = -14$. The factors of $a_3 = 1$ are $\pm 1$; the factors of $a_0 = -14$ are $\pm 1, \pm 2, \pm 7$. By Theorem 4.5.F, Rational Zeros Theorem, the possible rational zeros of $f$ are $p/q$ where $p$ is a factor of $a_0 = -14$ and $q$ is a factor $a_3 = 1$. So the possible rational zeros of $f$ are $\pm 1, \pm 2, \pm 7$. We have $f(1) = -10$, $f(-1) = -18$, $f(2) = 0$, $f(-2) = -28$, $f(7) = 350$, and $f(-7) = -378$. So $x = 2$ is a zero of $x^3 + 3x - 14$ and by the Factor Theorem, Theorem 4.5.C, $x - 2$ is a factor of $x^3 + 3x - 14$.





## Cubic Formula Problem 8 (continued)

**Solution (continued).** So we divide $x^3 + 3x - 14$ by $x - 2$:

$$
\begin{array}{r|rrrrrrr}
 & & & x^2 & + & 2x & + & 7 \\
\hline
x-2 & x^3 & & & + & 3x & + & 7 \\
 & x^3 & - & 2x^2 & & & & \\
\hline
 & & & 2x^2 & + & 3x & & \\
 & & & 2x^2 & - & 4x & & \\
\hline
 & & & & & 7x & - & 14 \\
 & & & & & 7x & - & 14 \\
\hline
 & & & & & & & 0
\end{array}
$$

So we have that $x^3 + 3x - 14 = (x^2 + 2x + 7)(x - 2)$. Notice that for $x^2 + 2x + 7$ that the discriminant is $b^2 - 4ac = (2)^2 - 4(1)(7) = -24 < 0$, so there are no real zeros of $x^2 + 2x + 7$ and hence no other zeros of $x^3 + 3x - 14$. That is, the only solution to $x^3 + 3x - 14 = 0$ is $\boxed{x = 2}$. □