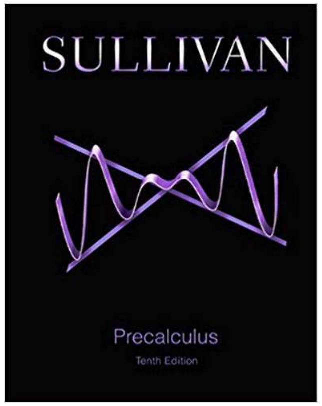
Precalculus 1 (Algebra)

Chapter 4. Polynomial and Rational Functions

4.5. The Real Zeros of a Polynomial Function—Exercises, Examples, Proofs



Theorem 4.5.B. Remainder Theorem

Theorem 4.5.B. Remainder Theorem. Let f be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

Proof. Let $g(x) = x - c$. Then by the Division Algorithm for Polynomials,Theorem 4.5.A, $f(x)/g(x) = q(x) + r(x)/g(x)$ where the remainder $r(x)$ is either the zero polynomial or a polynomial of degree less than that of $g(x)$. Since g is of degree 1, then $r(x)$ must either be the zero polynomial or of degree 0; that is, $r(x)$ is some constant, say $r(x) = R$. So the Division Algorithm reduces to $f(x)/(x - c) = q(x) + R/(x - c)$ (for $x \neq c$) or $f(x) = (x - c)q(x) + R$ (for all $x \in \mathbb{R}$). So with $x = c$ we have $f(c) = ((c) - c)q(c) + R = 0 + R = R$. That is, $f(x) = (x - c)q(x) + f(c)$ and the remainder when f is divided by $x - c$ is $f(c)$, as claimed. □

Theorem 4.5.C. Factor Theorem

Theorem 4.5.C. Factor Theorem. Let f be a polynomial function. Then $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$.

Proof. First, suppose that $f(c) = 0$. Then by the Remainder Theorem, Theorem 4.5.B, $f(x) = (x - c)q(x) + f(c) = (x - c)q(x) + 0 = (x - c)q(x)$ for some polynomial q. That is, $x - c$ is a factor of f, as claimed.

Second, suppose that $x - c$ is a factor of f. Then there is a polynomial q such that $f(x) = (x - c)q(x)$. With $x = c$ we then have $f(c) = ((c) - c)q(c) = (0)q(c) = 0$, as claimed. □

Page 233 Number 18

Page 233 Number 18. Consider $f(x) = x^6 - 16x^4 + x^2 - 16$ and $x - c = x - (-4) = x + 4$. Use the Remainder Theorem to find the remainder when $f(x)$ is divided by $x - c = x + 4$. Then use the Factor Theorem to determine whether $x - c$ is a factor of $f(x)$.

Solution. By the Remainder Theorem, Theorem 4.5.B, if polynomial function f is divided by $x - c$ then the remainder is $f(c)$. So if we divide $f(x) = x^6 - 16x^4 + x^2 - 16$ by $x + 4 = x - (-4)$ then the remainder is $f(-4) = (-4)^6 - 16(-4)^4 + (-4)^2 - 16 = \boxed{0}$. By the Factor Theorem, Theorem 4.5.C, $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$. With $c = -4$ we see that $f(-4) = 0$ and so $\boxed{x - (-4) = x + 4 \text{ is a factor of } f}$. □

Theorem 4.5.D. Number of Real Zeros

Theorem 4.5.D. Number of Real Zeros. A polynomial function cannot have more real zeros than its degree.

Proof. By the Factor Theorem, Theorem 4.5.C, if r is a real zero of a polynomial function f, then $f(r) = 0$, and $x - r$ is a factor of f. So each real zero r of f corresponds to a factor $x - r$ of degree 1. If we multiply all first degree factors of f together, then we get a factor of f which has degree equal to the number of first degree factors. But the degree of a polynomial factor of f cannot exceed the degree of f, so the number of first degree factors of f (and hence the number of real zeros of f) cannot exceed the degree of f, as claimed. □

Page 233 Number 30

Page 233 Number 30. Consider $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$. Give the maximum number of real zeros that each polynomial function may have. Then use Descartes' Rule of Signs to determine how many positive and how many negative zeros each polynomial function may have. Do not attempt to find the zeros.

Solution. Since f is a 5th degree polynomial function, then by Theorem 4.5.D, Number of Zeros, it has a $\boxed{\text{maximum of 5 real zeros}}$. The coefficients are (omitting coefficients of 0, if present) are: $1, -1, 1, -1, 1, -1$ so there are 5 sign changes and hence by Descartes' Rule of Signs, Theorem 4.5.E, the $\boxed{\text{number of positive zeros of } f \text{ is 5, 3, or 1}}$. Now $f(-x) = (-x)^5 - (-x)^4 + (-x)^3 - (-x)^2 + (-x) - 1 = -x^5 - x^4 - x^3 - x^2 - x - 1$ and the coefficients are: $-1, -1, -1, -1, -1, -1$ so there are 0 sign changes and hence by Descartes' Rule of Signs, Theorem 4.5.D, the $\boxed{\text{number of negative zeros of } f \text{ is } 0}$. □

Page 233 Number 42

Page 233 Number 42. List the potential rational zeros of $f(x) = 3x^5 - x^2 + 2x + 18$. Do not attempt to find the zeros.

Solution. We find the factors of the leading coefficient $a_5 = 3$ and the constant term $a_0 = 18$. The factors of $a_5 = 3$ are ± 1 and ± 3; the factors of $a_0 = 18$ are $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9 \pm 18$. By Theorem 4.5.F, Rational Zeros Theorem, the possible rational zeros of f are p/q where p is a factor of $a_0 = 18$ and q is a factor $a_5 = 3$. So the possible rational zeros of f are $\boxed{\pm 1, \pm 2, \pm 3, \pm 6, \pm 9 \pm 18, \pm 1/3, \text{ and } \pm 2/3}$. □

Page 233 Number 54

Page 233 Number 54. Use the Rational Zeros Theorem to find all the real zeros of $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$. Use the zeros to factor f over the real numbers.

Solution. We find the factors of the leading coefficient $a_4 = 1$ and the constant term $a_0 = 8$. The factors of $a_4 = 1$ are ± 1; the factors of $a_0 = 8$ are $\pm 1, \pm 2, \pm 4, \pm 8$. By Theorem 4.5.F, Rational Zeros Theorem, the possible rational zeros of f are p/q where p is a factor of $a_0 = 8$ and q is a factor $a_4 = 1$. So the possible rational zeros of f are $\pm 1, \pm 2, \pm 4, \pm 8$. We use the Factor Theorem, Theorem 4.5.C, to test these possible zeros. We have $f(1) = 6$, $f(-1) = 0$, $f(2) = 0$, $f(-2) = 0$, $f(4) = 120$, $f(-4) = 217$, $f(8) = 3240$, and $f(-8) = 4200$. So by the Factor Theorem, $x - (-1) = x + 1$, $x - 2$, and $x - (-2) = x + 2$ are factors of f. Hence the product
$(x+1)(x-2)(x+2) = (x+1)(x^2-4) = x^3 + x^2 - 4x - 4$ is a factor of f.

Page 233 Number 54 (continued)

Solution (continued). Since f is a degree 4 polynomial function, then there must be another first degree factor. (Also, this factor will yield another zero of f; it is unlikely to involve an irrational number so we might suspect that one of the known zeros is of multiplicity 2.) We perform long division to find this factor.

$$
\begin{array}{r|rrrrrrrrr}
 & & & & & & & x & - & 2 \\
\hline
x^3+x^2-4x-4 & x^4 & - & x^3 & - & 6x^2 & + & 4x & + & 8 \\
 & x^4 & + & x^3 & - & 4x^2 & - & 4x & & \\
\hline
 & & - & 2x^3 & - & 2x^2 & + & 8x & + & 8 \\
 & & - & 2x^3 & - & 2x^2 & + & 8x & + & 8 \\
\hline
 & & & & & & & & & 0
\end{array}
$$

So $x-2$ divides f (since the remainder is 0) and $x-2$ is the missing factor. We can factor f as $\boxed{f(x)=(x+1)(x-2)^2(x+2)}$. □

Theorem 4.5.I. Bounds on Zeros

Theorem 4.5.I. Bounds on Zeros. Let f denote a polynomial function whose leading coefficient is positive.

If $M>0$ is a real number and $f(x)=(x-M)q(x)+R$ where the coefficients of q are nonnegative and remainder R is nonnegative, then M is an upper bound to the zeros of f.

If $m<0$ is a real number and $f(x)=(x-m)q(x)+R$ where the coefficients of q (in standard form) followed by R alternate positive (or 0) and negative (or 0), then m is a lower bound to the zeros of f.

Proof. Suppose M is a positive real number such that $f(x)=(x-M)q(x)+R$ where the coefficients of q are nonnegative and remainder R is nonnegative. Then for $x>M$, $x-M>0$ and since $x>M>0$ then $q(x)>0$ so that $f(x)=(x-M)q(x)+R>0$. Hence, $f(x)$ cannot be 0 for $x>M$ and M is an upper bound to the zeros of f, as claimed.

Theorem 4.5.I (continued)

Proof (continued). Suppose $m<0$ is a real number and $f(x)=(x-m)q(x)+R$ where the coefficients of q (in standard form) followed by R alternate positive (or 0) and negative (or 0). Then for $x<m$, $x-m<0$ and since $x<m<0$ then $q(x)=b_mx^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0$, where $b_m\neq 0$, satisfies:
(1) If m is even and $b_m>0$ then $q(x)>0$, $b_0\geq 0$, $R\leq 0$, and so $f(x)=(x-m)q(x)+R<0$,
(2) if m is odd and $b_m>0$ then $q(x)<0$, $b_0\leq 0$, $R\geq 0$, and so $f(x)=(x-m)q(x)+R>0$,
(3) if m is even and $b_m<0$ then $q(x)<0$, $b_0\leq 0$, $R\geq 0$, and so $f(x)=(x-m)q(x)+R>0$, and
(4) if m is odd and $b_m<0$ then $q(x)>0$, $b_0\geq 0$, $R\leq 0$, and so $f(x)=(x-m)q(x)+R<0$.
Hence, $f(x)$ cannot be 0 for $x<m$ and m is a lower bound to the zeros of f, as claimed. □

Page 234 Number 70

Page 234 # 70. Find bounds on real zeros of $f(x)=x^4-5x^2-36$.
Solution. This requires potentially a large amount of long division (the text approaches this using the shortcut of "synthetic division," which is covered in Appendix A.4). We just present the division that yields the bounds. Consider the case $M=3$. We divide f by $x-M=x-3$:

$$
\begin{array}{r|rrrrrrr}
 & & & x^3 & + & 3x^2 & + & 4x & + & 12 \\
\hline
x-3 & x^4 & & & - & 5x^2 & & & - & 36 \\
 & x^4 & - & 3x^3 & & & & & & \\
\hline
 & & & 3x^3 & - & 5x^2 & & & & \\
 & & & 3x^3 & - & 9x^2 & & & & \\
\hline
 & & & & & 4x^2 & & & & \\
 & & & & & 4x^2 & - & 12x & & \\
\hline
 & & & & & & & 12x & - & 36 \\
 & & & & & & & 12x & - & 36 \\
\hline
 & & & & & & & & & 0
\end{array}
$$

Page 234 Number 70 (continued 1)

Page 234 Number 70. Find bounds on the real zeros of $f(x) = x^4 - 5x^2 - 36$.

Solution (continued). In the equation $f(x) = (x-3)q(x) + R$ we have $q(x) = x^3 + 3x^2 + 4x + 12$ and $R = 0$. So the coefficients of q are nonnegative and R is nonnegative. By Theorem 4.5.I, Bounds on Zeros, $\boxed{M = 3 \text{ is an upper bound on the zeros of } f}$. We can verify that 2 is *not* an upper bound on the zeros of f. ...

Page 234 Number 70 (continued 2)

Page 234 Number 70. Find bounds on the real zeros of $f(x) = x^4 - 5x^2 - 36$.

Solution (continued). Next, consider the case $m = -3$. We divide f by $x - m = x - (-3) = x + 3$:

$$
\begin{array}{r|rrrrrrrr}
 & & & x^3 & - & 3x^2 & + & 4x & - & 12 \\
\hline
x+3 & x^4 & & & - & 5x^2 & & & - & 36 \\
 & x^4 & + & 3x^3 & & & & & & \\
\hline
 & & - & 3x^3 & - & 5x^2 & & & & \\
 & & - & 3x^3 & - & 9x^2 & & & & \\
\hline
 & & & & & 4x^2 & & & & \\
 & & & & & 4x^2 & + & 12x & & \\
\hline
 & & & & & & - & 12x & - & 36 \\
 & & & & & & - & 12x & - & 36 \\
\hline
 & & & & & & & & & 0
\end{array}
$$

Page 234 Number 70 (continued 2)

Page 234 Number 70. Find bounds on the real zeros of $f(x) = x^4 - 5x^2 - 36$.

Solution (continued). In the equation $f(x) = (x+3)q(x) + R$ we have $q(x) = x^3 - 3x^2 + 4x - 12$ and $R = 0$. So the coefficients of q are $1, -3, 4, -12$ and $R = 0$. Hence the coefficients of q (in standard form) followed by R alternate positive (or 0) and negative (or 0). By Theorem 4.5.I, Bounds on Zeros, $\boxed{m = -3 \text{ is a lower bound on the zeros of } f}$. We can verify that -2 is *not* a lower bound on the zeros of f.

Therefore, the zeros of f are in the interval $\boxed{[m, M] = [-3, 3]}$. □

Page 234 Number 80

Page 234 Number 80. Use the Intermediate Value Theorem to show that $f(x) = x^4 + 8x^3 - x^2 + 2$ has a zero in the interval $[-1, 0]$.

Solution. Notice that $f(-1) = (-1)^4 + 8(-1)^3 - (-1)^2 + 2 = -6 < 0$ and $f(0) = (0)^4 + 8(0)^3 - (0)^2 + 2 = 2 > 0$. By the Intermediate Value Theorem, Theorem 4.5.J, for a polynomial function f if $a < b$ and if $f(a)$ and $f(b)$ are opposite sign, then there is at least one real zero of f between a and b. So with $a = -1$, $b = 0$, we have $f(a) = f(-1) = -6 < 0$ and $f(b) = f(0) = 2 > 0$, and the Intermediate Value Theorem implies that f has a zero between $a = -1$ and $b = 0$. That is, f has a zero in $[-1, 0]$. □