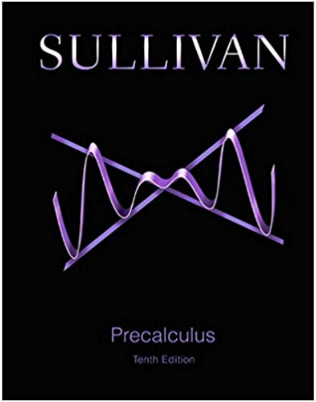
Precalculus 1 (Algebra)

Chapter 4. Polynomial and Rational Functions

4.5. The Real Zeros of a Polynomial Function—Exercises, Examples, Proofs



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Theorem 4.5.B. Remainder Theorem

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Proof. Let $g(x) = x - c$. Then by the Division Algorithm for Polynomials,Theorem 4.5.A, $f(x)/g(x) = q(x) + r(x)/g(x)$ where the remainder $r(x)$ is either the zero polynomial or a polynomial of degree less than that of $g(x)$. Since g is of degree 1, then $r(x)$ must either be the zero polynomial or of degree 0; that is, $r(x)$ is some constant, say $r(x) = R$.

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Theorem 4.5.C. Factor Theorem

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Proof. First, suppose that $f(c) = 0$. Then by the Remainder Theorem, Theorem 4.5.B,
$f(x) = (x - c)q(x) + f(c) = (x - c)q(x) + 0 = (x - c)q(x)$ for some polynomial q. That is, $x - c$ is a factor of f, as claimed.

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Second, suppose that $x - c$ is a factor of f. Then there is a polynomial q such that $f(x) = (x - c)q(x)$. With $x = c$ we then have
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Solution. By the Remainder Theorem, Theorem 4.5.B, if polynomial function f is divided by $x - c$ then the remainder is $f(c)$. So if we divide $f(x) = x^6 - 16x^4 + x^2 - 16$ by $x + 4 = x - (-4)$ then the remainder is $f(-4) = (-4)^6 - 16(-4)^4 + (-4)^2 - 16 = \boxed{0}$.

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Theorem 4.5.D. Number of Real Zeros

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Proof. By the Factor Theorem, Theorem 4.5.C, if r is a real zero of a polynomial function f, then $f(r) = 0$, and $x - r$ is a factor of f. So each real zero r of f corresponds to a factor $x - r$ of degree 1. If we multiply all first degree factors of f together, then we get a factor of f which has degree equal to the number of first degree factors.

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Page 233 Number 30. Consider $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$. Give the maximum number of real zeros that each polynomial function may have. Then use Descartes' Rule of Signs to determine how many positive and how many negative zeros each polynomial function may have. Do not attempt to find the zeros.

Solution. Since f is a 5th degree polynomial function, then by Theorem 4.5.D, Number of Zeros, it has a $\boxed{\text{maximum of 5 real zeros}}$. The coefficients are (omitting coefficients of 0, if present) are: $1, -1, 1, -1, 1, -1$ so there are 5 sign changes and hence by Descartes' Rule of Signs, Theorem 4.5.E, the $\boxed{\text{number of positive zeros of } f \text{ is 5, 3, or 1}}$.

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Page 233 Number 42

Page 233 Number 42. List the potential rational zeros of $f(x) = 3x^5 - x^2 + 2x + 18$. Do not attempt to find the zeros.

Solution. We find the factors of the leading coefficient $a_5 = 3$ and the constant term $a_0 = 18$. The factors of $a_5 = 3$ are ± 1 and ± 3; the factors of $a_0 = 18$ are $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9 \pm 18$.

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Page 233 Number 54

Page 233 Number 54. Use the Rational Zeros Theorem to find all the real zeros of $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$. Use the zeros to factor f over the real numbers.

Solution. We find the factors of the leading coefficient $a_4 = 1$ and the constant term $a_0 = 8$. The factors of $a_4 = 1$ are ± 1; the factors of $a_0 = 8$ are $\pm 1, \pm 2, \pm 4, \pm 8$.

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Page 233 Number 54 (continued)

Solution (continued). Since f is a degree 4 polynomial function, then there must be another first degree factor. (Also, this factor will yield another zero of f; it is unlikely to involve an irrational number so we might suspect that one of the known zeros is of multiplicity 2.) We perform long division to find this factor.

$$
\begin{array}{r|rrrrr}
 & & & & x & -\,2 \\
\hline
x^3+x^2-4x-4 & x^4 & -\,x^3 & -\,6x^2 & +\,4x & +\,8 \\
 & x^4 & +\,x^3 & -\,4x^2 & -\,4x & \\
\hline
 & & -\,2x^3 & -\,2x^2 & +\,8x & +\,8 \\
 & & -\,2x^3 & -\,2x^2 & +\,8x & +\,8 \\
\hline
 & & & & & 0
\end{array}
$$

Page 233 Number 54 (continued)

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$$\begin{array}{r|l} & x - 2 \\ \hline x^3 + x^2 - 4x - 4 & x^4 - x^3 - 6x^2 + 4x + 8 \\ & x^4 + x^3 - 4x^2 - 4x \\ \hline & -2x^3 - 2x^2 + 8x + 8 \\ & -2x^3 - 2x^2 + 8x + 8 \\ \hline & 0 \end{array}$$

So $x - 2$ divides f (since the remainder is 0) and $x - 2$ is the missing factor. We can factor f as $\boxed{f(x) = (x+1)(x-2)^2(x+2)}$. □

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$$
\begin{array}{r|rrrrrrrrr}
 & & & & & & & x & - & 2 \\
\hline
x^3 + x^2 - 4x - 4 & x^4 & - & x^3 & - & 6x^2 & + & 4x & + & 8 \\
 & x^4 & + & x^3 & - & 4x^2 & - & 4x & & \\
\hline
 & & - & 2x^3 & - & 2x^2 & + & 8x & + & 8 \\
 & & - & 2x^3 & - & 2x^2 & + & 8x & + & 8 \\
\hline
 & & & & & & & & & 0
\end{array}
$$

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Theorem 4.5.I. Bounds on Zeros

Theorem 4.5.I. Bounds on Zeros. Let f denote a polynomial function whose leading coefficient is positive.

If $M > 0$ is a real number and $f(x) = (x - M)q(x) + R$ where the coefficients of q are nonnegative and remainder R is nonnegative, then M is an upper bound to the zeros of f.

If $m < 0$ is a real number and $f(x) = (x - m)q(x) + R$ where the coefficients of q (in standard form) followed by R alternate positive (or 0) and negative (or 0), then m is a lower bound to the zeros of f.

Proof. Suppose M is a positive real number such that $f(x) = (x - M)q(x) + R$ where the coefficients of q are nonnegative and remainder R is nonnegative. Then for $x > M$, $x - M > 0$ and since $x > M > 0$ then $q(x) > 0$ so that $f(x) = (x - M)q(x) + R > 0$. Hence, $f(x)$ cannot be 0 for $x > M$ and M is an upper bound to the zeros of f, as claimed.

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Proof. Suppose M is a positive real number such that $f(x) = (x - M)q(x) + R$ where the coefficients of q are nonnegative and remainder R is nonnegative. Then for $x > M$, $x - M > 0$ and since $x > M > 0$ then $q(x) > 0$ so that $f(x) = (x - M)q(x) + R > 0$. Hence, $f(x)$ cannot be 0 for $x > M$ and M is an upper bound to the zeros of f, as claimed.

Theorem 4.5.I (continued)

Proof (continued). Suppose $m < 0$ is a real number and $f(x) = (x - m)q(x) + R$ where the coefficients of q (in standard form) followed by R alternate positive (or 0) and negative (or 0). Then for $x < m$, $x - m < 0$ and since $x < m < 0$ then
$q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$, where $b_m \neq 0$, satisfies:
(1) If m is even and $b_m > 0$ then $q(x) > 0$, $b_0 \geq 0$, $R \leq 0$, and so $f(x) = (x - m)q(x) + R < 0$,

Theorem 4.5.I (continued)

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(1) If m is even and $b_m > 0$ then $q(x) > 0$, $b_0 \geq 0$, $R \leq 0$, and so $f(x) = (x - m)q(x) + R < 0$,

(2) if m is odd and $b_m > 0$ then $q(x) < 0$, $b_0 \leq 0$, $R \geq 0$, and so $f(x) = (x - m)q(x) + R > 0$,

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(2) if m is odd and $b_m > 0$ then $q(x) < 0$, $b_0 \leq 0$, $R \geq 0$, and so $f(x) = (x - m)q(x) + R > 0$,

(3) if m is even and $b_m < 0$ then $q(x) < 0$, $b_0 \leq 0$, $R \geq 0$, and so $f(x) = (x - m)q(x) + R > 0$, and

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(2) if m is odd and $b_m > 0$ then $q(x) < 0$, $b_0 \leq 0$, $R \geq 0$, and so $f(x) = (x - m)q(x) + R > 0$,
(3) if m is even and $b_m < 0$ then $q(x) < 0$, $b_0 \leq 0$, $R \geq 0$, and so $f(x) = (x - m)q(x) + R > 0$, and
(4) if m is odd and $b_m < 0$ then $q(x) > 0$, $b_0 \geq 0$, $R \leq 0$, and so $f(x) = (x - m)q(x) + R < 0$.
Hence, $f(x)$ cannot be 0 for $x < m$ and m is a lower bound to the zeros of f, as claimed. □

Theorem 4.5.I (continued)

Proof (continued). Suppose $m < 0$ is a real number and $f(x) = (x - m)q(x) + R$ where the coefficients of q (in standard form) followed by R alternate positive (or 0) and negative (or 0). Then for $x < m$, $x - m < 0$ and since $x < m < 0$ then $q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$, where $b_m \neq 0$, satisfies:
(1) If m is even and $b_m > 0$ then $q(x) > 0$, $b_0 \geq 0$, $R \leq 0$, and so $f(x) = (x - m)q(x) + R < 0$,
(2) if m is odd and $b_m > 0$ then $q(x) < 0$, $b_0 \leq 0$, $R \geq 0$, and so $f(x) = (x - m)q(x) + R > 0$,
(3) if m is even and $b_m < 0$ then $q(x) < 0$, $b_0 \leq 0$, $R \geq 0$, and so $f(x) = (x - m)q(x) + R > 0$, and
(4) if m is odd and $b_m < 0$ then $q(x) > 0$, $b_0 \geq 0$, $R \leq 0$, and so $f(x) = (x - m)q(x) + R < 0$.
Hence, $f(x)$ cannot be 0 for $x < m$ and m is a lower bound to the zeros of f, as claimed. □

Page 234 Number 70

Page 234 # 70. Find bounds on real zeros of $f(x) = x^4 - 5x^2 - 36$.

Solution. This requires potentially a large amount of long division (the text approaches this using the shortcut of "synthetic division," which is covered in Appendix A.4). We just present the division that yields the bounds.

Page 234 Number 70

Page 234 # 70. Find bounds on real zeros of $f(x) = x^4 - 5x^2 - 36$.
Solution. This requires potentially a large amount of long division (the text approaches this using the shortcut of "synthetic division," which is covered in Appendix A.4). We just present the division that yields the bounds. Consider the case $M = 3$. We divide f by $x - M = x - 3$:

$$
\begin{array}{r|rrrrrrrrr}
 & & & x^3 & + & 3x^2 & + & 4x & + & 12 \\
\hline
x-3 & x^4 & & & - & 5x^2 & & & - & 36 \\
 & x^4 & - & 3x^3 & & & & & & \\
\hline
 & & & 3x^3 & - & 5x^2 & & & & \\
 & & & 3x^3 & - & 9x^2 & & & & \\
\hline
 & & & & & 4x^2 & & & & \\
 & & & & & 4x^2 & - & 12x & & \\
\hline
 & & & & & & & 12x & - & 36 \\
 & & & & & & & 12x & - & 36 \\
\hline
 & & & & & & & & & 0
\end{array}
$$

Page 234 Number 70

Page 234 # 70. Find bounds on real zeros of $f(x) = x^4 - 5x^2 - 36$.
Solution. This requires potentially a large amount of long division (the text approaches this using the shortcut of "synthetic division," which is covered in Appendix A.4). We just present the division that yields the bounds. Consider the case $M = 3$. We divide f by $x - M = x - 3$:

$$
\begin{array}{r|l}
 & x^3 + 3x^2 + 4x + 12 \\
\hline
x - 3 & x^4 \quad\quad\quad - 5x^2 \quad\quad - 36 \\
 & x^4 - 3x^3 \\
\hline
 & \quad 3x^3 - 5x^2 \\
 & \quad 3x^3 - 9x^2 \\
\hline
 & \quad\quad 4x^2 \\
 & \quad\quad 4x^2 - 12x \\
\hline
 & \quad\quad\quad 12x - 36 \\
 & \quad\quad\quad 12x - 36 \\
\hline
 & \quad\quad\quad\quad 0
\end{array}
$$

Page 234 Number 70 (continued 1)

Page 234 Number 70. Find bounds on the real zeros of $f(x) = x^4 - 5x^2 - 36$.

Solution (continued). In the equation $f(x) = (x - 3)q(x) + R$ we have $q(x) = x^3 + 3x^2 + 4x + 12$ and $R = 0$. So the coefficients of q are nonnegative and R is nonnegative. By Theorem 4.5.I, Bounds on Zeros, $\boxed{M = 3 \text{ is an upper bound on the zeros of } f}$. We can verify that 2 is *not* an upper bound on the zeros of f. . . .

Page 234 Number 70 (continued 2)

Page 234 Number 70. Find bounds on the real zeros of $f(x) = x^4 - 5x^2 - 36$.

Solution (continued). Next, consider the case $m = -3$. We divide f by $x - m = x - (-3) = x + 3$:

$$
\begin{array}{r|rrrrrrrrr}
 & & & x^3 & - & 3x^2 & + & 4x & - & 12 \\
\hline
x+3 & x^4 & & & - & 5x^2 & & & - & 36 \\
 & x^4 & + & 3x^3 & & & & & & \\
\hline
 & & - & 3x^3 & - & 5x^2 & & & & \\
 & & - & 3x^3 & - & 9x^2 & & & & \\
\hline
 & & & & & 4x^2 & & & & \\
 & & & & & 4x^2 & + & 12x & & \\
\hline
 & & & & & & - & 12x & - & 36 \\
 & & & & & & - & 12x & - & 36 \\
\hline
 & & & & & & & & & 0
\end{array}
$$

Page 234 Number 70 (continued 2)

Page 234 Number 70. Find bounds on the real zeros of $f(x) = x^4 - 5x^2 - 36$.

Solution (continued). In the equation $f(x) = (x+3)q(x) + R$ we have $q(x) = x^3 - 3x^2 + 4x - 12$ and $R = 0$. So the coefficients of q are $1, -3, 4, -12$ and $R = 0$. Hence the coefficients of q (in standard form) followed by R alternate positive (or 0) and negative (or 0). By Theorem 4.5.I, Bounds on Zeros, $\boxed{m = -3 \text{ is a lower bound on the zeros of } f}$. We can verify that -2 is *not* a lower bound on the zeros of f.

Therefore, the zeros of f are in the interval $\boxed{[m, M] = [-3, 3]}$. □

Page 234 Number 70 (continued 2)

Page 234 Number 70. Find bounds on the real zeros of $f(x) = x^4 - 5x^2 - 36$.

Solution (continued). In the equation $f(x) = (x+3)q(x) + R$ we have $q(x) = x^3 - 3x^2 + 4x - 12$ and $R = 0$. So the coefficients of q are $1, -3, 4, -12$ and $R = 0$. Hence the coefficients of q (in standard form) followed by R alternate positive (or 0) and negative (or 0). By Theorem 4.5.I, Bounds on Zeros, $\boxed{m = -3 \text{ is a lower bound on the zeros of } f}$. We can verify that -2 is *not* a lower bound on the zeros of f.

Therefore, the zeros of f are in the interval $\boxed{[m, M] = [-3, 3]}$. □

Page 234 Number 80

Page 234 Number 80. Use the Intermediate Value Theorem to show that $f(x) = x^4 + 8x^3 - x^2 + 2$ has a zero in the interval $[-1, 0]$.

Solution. Notice that $f(-1) = (-1)^4 + 8(-1)^3 - (-1)^2 + 2 = -6 < 0$ and $f(0) = (0)^4 + 8(0)^3 - (0)^2 + 2 = 2 > 0$. By the Intermediate Value Theorem, Theorem 4.5.J, for a polynomial function f if $a < b$ and if $f(a)$ and $f(b)$ are opposite sign, then there is at least one real zero of f between a and b.

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Page 234 Number 80

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Cubic Formula Problem 1

Cubic Formula Problem 1. Show that the general cubic equation $y^3 + by^2 + cy + d = 0$ can be transformed into an equation of the form $x^3 + px + q = 0$ by using the substitution $y = x - b/3$.

Solution. With $y = x - b/3$, we have

$$y^2 = (x - b/3)^2 = x^2 - 2bx/3 + b^2/9 \text{ and}$$

$$y^3 = (x - b/3)(x^2 - 2bx/3 + b^2/9) = x^3 - 2bx^2/3 + b^2x/9 - bx^2/3$$

$$+2b^2x/9 - b^3/27 = x^3 - 3bx^2/3 + 3b^2x/9 - b^3/27$$

$$= x^3 - bx^2 + b^2x/3 - b^3/27.$$

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$$y^3 = (x - b/3)(x^2 - 2bx/3 + b^2/9) = x^3 - 2bx^2/3 + b^2x/9 - bx^2/3$$
$$+2b^2x/9 - b^3/27 = x^3 - 3bx^2/3 + 3b^2x/9 - b^3/27$$
$$= x^3 - bx^2 + b^2x/3 - b^3/27.$$

We then have

$$0 = y^3 + by^2 + cy + d = (x^3 - bx^2 + b^2x/3 - b^3/27) + b(x^2 - 2bx/3 + b^2/9)$$
$$+c(x - b/3) + d = x^3 + (-b^2/3 + c)x + (2b^3/27 - bc/3 + d).$$

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$$= x^3 - bx^2 + b^2x/3 - b^3/27.$$

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$$+c(x - b/3) + d = x^3 + (-b^2/3 + c)x + (2b^3/27 - bc/3 + d).$$

Cubic Formula Problem 1 (continued)

Cubic Formula Problem 1. Show that the general cubic equation $y^3 + by^2 + cy + d = 0$ can be transformed into an equation of the form $x^3 + px + q = 0$ by using the substitution $y = x - b/3$.

Solution. ...

$$0 = y^3 + by^2 + cy + d = x^3 + (-b^2/3 + c)x + (2b^3/27 - bc/3 + d).$$

We then have $x^3 + px + q = 0$ where $p = c - b^2/3$ and $q = 2b^3/27 - bc/3 + d$. □

Cubic Formula Problem 2

Cubic Formula Problem 2. In the equation $x^3 + px + q = 0$, replace x by $H + K$. Let $3HK = -p$, and show that $H^3 + K^3 = -q$.

Solution. With $x = H + K$ we have

$$x^3 = (H+K)^3 = (H+K)(H^2+2HK+K^2) = H^3 + 2H^2K + HK^2 + H^2K$$

$$+2HK^2 + K^3 = H^3 + 3H^2K + 3HK^2 + K^3.$$

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$$x^3 = (H+K)^3 = (H+K)(H^2 + 2HK + K^2) = H^3 + 2H^2K + HK^2 + H^2K$$
$$+2HK^2 + K^3 = H^3 + 3H^2K + 3HK^2 + K^3.$$

With $3HK = -p$, or $p = -3HK$ we then have

$$0 = x^3 + px + q = (H^3 + 3H^2K + 3HK^2 + K^3) + (-3HK)(H+K) + q$$
$$= H^3 + 3H^2K + 3HK^2 + K^3 - 3H^2K - 3HK^2 + q = H^3 + K^3 + q$$

or $H^3 + K^3 = -q$, as claimed. □

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Cubic Formula Problem 2. In the equation $x^3 + px + q = 0$, replace x by $H + K$. Let $3HK = -p$, and show that $H^3 + K^3 = -q$.

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$$= H^3 + 3H^2K + 3HK^2 + K^3 - 3H^2K - 3HK^2 + q = H^3 + K^3 + q$$

or $H^3 + K^3 = -q$, as claimed. □

Cubic Formula Problem 3

Cubic Formula Problem 3. Based on Cubic Formula Problem 2, we have two equations $3HK = -p$ and $H^3 + K^3 = -q$. Solve for K in $3HK = -p$ and substitute into $H^3 + K^3 = -q$. Then show that $H = \sqrt[3]{\dfrac{-q}{2} \pm \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}}$. HINT: Look for an equation that is quadratic in form.

Solution. Since $3HK = -p$, then $K = -p/(3H)$. So $H^3 + K^3 = -q$ implies that $-q = H^3 + (-p/(3H))^3 = H^3 - p^3/(27H^3)$ or, multiplying both sides of this last equation by H^3, $-qH^3 = H^6 - p^3/27$ or $H^6 + qH^3 - p^3/27 = 0$ or $(H^3)^2 + q(H^3) - p^3/27 = 0$. So we have a quadratic equation in the unknown H^3 and so we can solve for H^3 using the quadratic formula to get:

$$H^3 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(q) \pm \sqrt{(q)^2 - 4(1)(-p^3/27)}}{2(1)} = \dots$$

Cubic Formula Problem 3

Cubic Formula Problem 3. Based on Cubic Formula Problem 2, we have two equations $3HK = -p$ and $H^3 + K^3 = -q$. Solve for K in $3HK = -p$ and substitute into $H^3 + K^3 = -q$. Then show that $H = \sqrt[3]{\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$. HINT: Look for an equation that is quadratic in form.

Solution. Since $3HK = -p$, then $K = -p/(3H)$. So $H^3 + K^3 = -q$ implies that $-q = H^3 + (-p/(3H))^3 = H^3 - p^3/(27H^3)$ or, multiplying both sides of this last equation by H^3, $-qH^3 = H^6 - p^3/27$ or $H^6 + qH^3 - p^3/27 = 0$ or $(H^3)^2 + q(H^3) - p^3/27 = 0$. So we have a quadratic equation in the unknown H^3 and so we can solve for H^3 using the quadratic formula to get:

$$H^3 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(q) \pm \sqrt{(q)^2 - 4(1)(-p^3/27)}}{2(1)} = \dots$$

Cubic Formula Problem 3 (continued)

Solution (continued). ...

$$H^3 = \frac{-q \pm \sqrt{q^2 + 4p^3/27}}{2} = \frac{-q}{2} \pm \frac{\sqrt{q^2 + 4p^3/27}}{2}$$

$$= \frac{-q}{2} \pm \frac{\sqrt{q^2 + 4p^3/27}}{\sqrt{4}} = \frac{-q}{2} \pm \sqrt{\frac{q^2 + 4p^3/27}{4}}$$

$$= \frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

Hence,

$$H = \sqrt[3]{\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

□

Cubic Formula Problem 4

Cubic Formula Problem 4. Use the solution for H from Cubic Formula Problem 3 and the equation $H^3 + K^3 = -q$ to show that $K = \sqrt[3]{\dfrac{-q}{2} \mp \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}}$.

Solution. With $H = \sqrt[3]{\dfrac{-q}{2} \pm \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}}$, we have

$$-q = H^3 + K^3 = \frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} + K^3 \text{ or}$$

$$K^3 = -q - \left(\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right) = \frac{-q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

So

$$K = \sqrt[3]{\frac{-q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}. \qquad \square$$

Cubic Formula Problem 4

Cubic Formula Problem 4. Use the solution for H from Cubic Formula Problem 3 and the equation $H^3 + K^3 = -q$ to show that

$$K = \sqrt[3]{\frac{-q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

Solution. With $H = \sqrt[3]{\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$, we have

$$-q = H^3 + K^3 = \frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} + K^3 \text{ or}$$

$$K^3 = -q - \left(\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right) = \frac{-q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

So

$$K = \sqrt[3]{\frac{-q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

□

Cubic Formula Problem 5

Cubic Formula Problem 5. Use the results from Cubic Formula Problems 2 to 4 to show that the solution of $x^3 + px + q = 0$ is

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

Solution. First, we have $x = H + K$, so

$$x = \sqrt[3]{\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

$$= \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

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Solution. First, we have $x = H + K$, so

$$x = \sqrt[3]{\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

$$= \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

Now with $x = H + K$, $3HK = -p$ (or $p = -3HK$), and $H^3 + K^3 = -q$ (or $q = -H^3 - K^3$) from Cubic Formula Problem 2, we have ...

Cubic Formula Problem 5

Cubic Formula Problem 5. Use the results from Cubic Formula Problems 2 to 4 to show that the solution of $x^3 + px + q = 0$ is

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

Solution. First, we have $x = H + K$, so

$$x = \sqrt[3]{\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

$$= \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

Now with $x = H + K$, $3HK = -p$ (or $p = -3HK$), and $H^3 + K^3 = -q$ (or $q = -H^3 - K^3$) from Cubic Formula Problem 2, we have . . .

Cubic Formula Problem 5 (continued)

Cubic Formula Problem 5. Use the results from Cubic Formula Problems 2 to 4 to show that the solution of $x^3 + px + q = 0$ is

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

Solution (continued). ...

$$x^3 + px + q = (H+K)^3 + (-3HK)(H+K) + (-H^3 - K^3)$$

$$= (H+K)(H^2 + 2HK + K^2) - 3H^2K - 3HK^2 - H^3 - K^3$$

$$= (H^3 + 2H^2K + HK^2 + H^2K + 2HK^2 + K^3) - 3H^2K - 3HK^2 - H^3 - K^3$$

$$= (H^3 + 3H^2K + 3HK^2 + K^3) - 3H^2K - 3HK^2 - H^3 - K^3 = 0,$$

as claimed. □

Cubic Formula Problem 6

Cubic Formula Problem 6. Use the result of Cubic Formula Problem 5 to solve the equation $x^3 - 6x - 9 = 0$.

Solution. Here, we have $p = -6$ and $q = -9$. Since by Cubic Problem 5,

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

we have (simplifying a little up front and skipping steps)

$$x = \sqrt[3]{\frac{9}{2} + \sqrt{\frac{81}{4} + \frac{-216}{27}}} + \sqrt[3]{\frac{9}{2} - \sqrt{\frac{81}{4} + \frac{-216}{27}}}$$

$$= \sqrt[3]{\frac{9}{2} + \sqrt{\frac{49}{4}}} + \sqrt[3]{\frac{9}{2} - \sqrt{\frac{49}{4}}} = \sqrt[3]{\frac{9}{2} + \frac{7}{2}} + \sqrt[3]{\frac{9}{2} - \frac{7}{2}}$$

$$= \sqrt[3]{8} + \sqrt[3]{1} = 2 + 1 = \boxed{3}.$$

□

Cubic Formula Problem 6

Cubic Formula Problem 6. Use the result of Cubic Formula Problem 5 to solve the equation $x^3 - 6x - 9 = 0$.

Solution. Here, we have $p = -6$ and $q = -9$. Since by Cubic Problem 5,

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

we have (simplifying a little up front and skipping steps)

$$x = \sqrt[3]{\frac{9}{2} + \sqrt{\frac{81}{4} + \frac{-216}{27}}} + \sqrt[3]{\frac{9}{2} - \sqrt{\frac{81}{4} + \frac{-216}{27}}}$$

$$= \sqrt[3]{\frac{9}{2} + \sqrt{\frac{49}{4}}} + \sqrt[3]{\frac{9}{2} - \sqrt{\frac{49}{4}}} = \sqrt[3]{\frac{9}{2} + \frac{7}{2}} + \sqrt[3]{\frac{9}{2} - \frac{7}{2}}$$

$$= \sqrt[3]{8} + \sqrt[3]{1} = 2 + 1 = \boxed{3}.$$

□

Cubic Formula Problem 7

Cubic Formula Problem 7. Use the result of Cubic Formula Problem 5 to solve $x^3 + 3x - 14 = 0$. Use a calculator to give a decimal approximation of the solution to the equation.

Solution. Here, we have $p = 3$ and $q = -14$. Since by Cubic Formula Problem 5,

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

we have (simplifying a little up front and skipping steps)

$$x = \sqrt[3]{\frac{14}{2} + \sqrt{\frac{196}{4} + \frac{27}{27}}} + \sqrt[3]{\frac{14}{2} - \sqrt{\frac{196}{4} + \frac{27}{27}}}$$

$$= \sqrt[3]{7 + \sqrt{49 + 1}} + \sqrt[3]{7 - \sqrt{49 + 1}} = \sqrt[3]{7 + \sqrt{50}} + \sqrt[3]{7 - \sqrt{50}} \ldots$$

Cubic Formula Problem 7

Cubic Formula Problem 7. Use the result of Cubic Formula Problem 5 to solve $x^3 + 3x - 14 = 0$. Use a calculator to give a decimal approximation of the solution to the equation.

Solution. Here, we have $p = 3$ and $q = -14$. Since by Cubic Formula Problem 5,

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we have (simplifying a little up front and skipping steps)

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$$= \sqrt[3]{7 + \sqrt{49 + 1}} + \sqrt[3]{7 - \sqrt{49 + 1}} = \sqrt[3]{7 + \sqrt{50}} + \sqrt[3]{7 - \sqrt{50}} \ldots$$

Cubic Formula Problem 7 (continued)

Cubic Formula Problem 7. Use the result of Cubic Formula Problem 5 to solve $x^3 + 3x - 14 = 0$. Use a calculator to give a decimal approximation of the solution to the equation.

Solution (continued). ...

$$x = \boxed{\sqrt[3]{7 + 5\sqrt{2}} + \sqrt[3]{7 - 5\sqrt{2}}},$$

since $\sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$. If we plug x into a calculator, we find that it simplifies to $\boxed{x = 2}$. □

Cubic Formula Problem 7 (continued)

Cubic Formula Problem 7. Use the result of Cubic Formula Problem 5 to solve $x^3 + 3x - 14 = 0$. Use a calculator to give a decimal approximation of the solution to the equation.

Solution (continued). ...

$$x = \boxed{\sqrt[3]{7 + 5\sqrt{2}} + \sqrt[3]{7 - 5\sqrt{2}}},$$

since $\sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$. If we plug x into a calculator, we find that it simplifies to $\boxed{x = 2}$. □

Cubic Formula Problem 8

Cubic Formula Problem 8. Use the methods of this section to solve the equation $x^3 + 3x - 14 = 0$.

Solution. We find the factors of the leading coefficient $a_3 = 1$ and the constant term $a_0 = -14$. The factors of $a_3 = 1$ are ± 1; the factors of $a_0 = -14$ are $\pm 1, \pm 2, \pm 7$.

Cubic Formula Problem 8

Cubic Formula Problem 8. Use the methods of this section to solve the equation $x^3 + 3x - 14 = 0$.

Solution. We find the factors of the leading coefficient $a_3 = 1$ and the constant term $a_0 = -14$. The factors of $a_3 = 1$ are ± 1; the factors of $a_0 = -14$ are $\pm 1, \pm 2, \pm 7$. By Theorem 4.5.F, Rational Zeros Theorem, the possible rational zeros of f are p/q where p is a factor of $a_0 = -14$ and q is a factor $a_3 = 1$. So the possible rational zeros of f are $\pm 1, \pm 2, \pm 7$.

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Cubic Formula Problem 8 (continued)

Solution (continued). So we divide $x^3 + 3x - 14$ by $x - 2$:

$$
\begin{array}{r|rrrrrrr}
 & & & x^2 & + & 2x & + & 7 \\
\hline
x - 2 & x^3 & & & + & 3x & + & 7 \\
 & x^3 & - & 2x^2 & & & & \\
\hline
 & & & 2x^2 & + & 3x & & \\
 & & & 2x^2 & - & 4x & & \\
\hline
 & & & & & 7x & - & 14 \\
 & & & & & 7x & - & 14 \\
\hline
 & & & & & & & 0
\end{array}
$$

So we have that $x^3 + 3x - 14 = (x^2 + 2x + 7)(x - 2)$. Notice that for $x^2 + 2x + 7$ that the discriminant is $b^2 - 4ac = (2)^2 - 4(1)(7) = -24 < 0$, so there are no real zeros of $x^2 + 2x + 7$ and hence no other zeros of $x^3 + 3x - 14$. That is, the only solution to $x^3 + 3x - 14 = 0$ is $\boxed{x = 2}$. □

Cubic Formula Problem 8 (continued)

Solution (continued). So we divide $x^3 + 3x - 14$ by $x - 2$:

$$
\begin{array}{rrrrrrrr}
 & & & x^2 & + & 2x & + & 7 \\
\cline{2-8}
x - 2 \big) & x^3 & & & + & 3x & + & 7 \\
 & x^3 & - & 2x^2 & & & & \\
\cline{2-8}
 & & & 2x^2 & + & 3x & & \\
 & & & 2x^2 & - & 4x & & \\
\cline{5-8}
 & & & & & 7x & - & 14 \\
 & & & & & 7x & - & 14 \\
\cline{5-8}
 & & & & & & & 0
\end{array}
$$

So we have that $x^3 + 3x - 14 = (x^2 + 2x + 7)(x - 2)$. Notice that for $x^2 + 2x + 7$ that the discriminant is $b^2 - 4ac = (2)^2 - 4(1)(7) = -24 < 0$, so there are no real zeros of $x^2 + 2x + 7$ and hence no other zeros of $x^3 + 3x - 14$. That is, the only solution to $x^3 + 3x - 14 = 0$ is $\boxed{x = 2}$. □