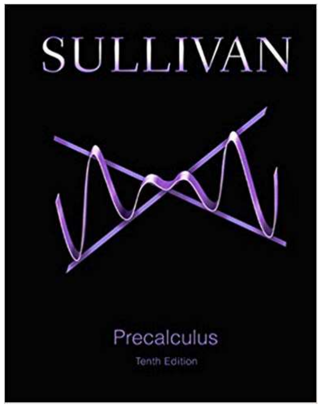
Precalculus 1 (Algebra)

Chapter 4. Polynomial and Rational Functions

4.6. Complex Zeros; Fundamental Theorem of Algebra—Exercises, Examples, Proofs



Theorem 4.6.B

Theorem 4.6.B. Every complex polynomial function f of degree $n \geq 1$ can be factored into n linear factors (not necessarily distinct) of the form

$$f(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_{n-1})(x - r_n)$$

where $a_n, r_1, r_2, \ldots, r_{n-1}, r_n$ are complex numbers. That is, every complex polynomial function of degree $n \geq 1$ has exactly n complex zeros, some of which may repeat.

Proof. Let

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0.$$

By the Fundamental Theorem of Algebra, Theorem 4.6.A, f has at least one zero, say r_1. Then by the Factor Theorem, Theorem 4.5.C, $x - r_1$ is a factor of f and $f(x) = (x - r_1)q_1(x)$ where q_1 is a complex polynomial of degree $n - 1$ whose leading coefficient is a_n. Repeating this argument n times (we are really using Mathematical Induction here; this is in the text book in Section 12.4), we get . . .

Theorem 4.6.B (continued)

Theorem 4.6.B. Every complex polynomial function f of degree $n \geq 1$ can be factored into n linear factors (not necessarily distinct) of the form

$$f(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_{n-1})(x - r_n)$$

where $a_n, r_1, r_2, \ldots, r_{n-1}, r_n$ are complex numbers. That is, every complex polynomial function of degree $n \geq 1$ has exactly n complex zeros, some of which may repeat.

Proof (continued).

$$f(x) = (x - r_1)(x - r_2)\cdots(x - r_n)q_n(x)$$

where q_n is a complex polynomial of degree $n - n = 0$ whose leading coefficient is also a_n. That is, $q_n(x) = a_nx^0 = a_n$ so that

$$f(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n).$$

Since f has exactly n linear factors, then it has exactly n (not necessarily distinct) zeros, as claimed. □

Theorem 4.6.C

Theorem 4.6.C. Conjugate Pairs Theorem. Let f be a polynomial function whose coefficients are real numbers. If $r = a + bi$ is a zero of f, then the complex conjugate $\overline{r} = a - bi$ is also a zero of f.

Proof. Let

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers and $a_n \neq 0$. If $r = a + bi$ is a zero of f, then $f(r) = f(a + bi) = 0$, so

$$a_nr^n + a_{n-1}r^{n-1} + \cdots + a_2r^2 + a_1r + a_0 = 0.$$

Then by properties of the conjugate (see Theorem A.7.A in A.7. Complex Numbers; Quadratic Equations in the Complex Number System) we have

$$\overline{a_nr^n + a_{n-1}r^{n-1} + \cdots + a_2r^2 + a_1r + a_0} = \overline{0}$$

$$\overline{a_nr^n} + \overline{a_{n-1}r^{n-1}} + \cdots + \overline{a_2r^2} + \overline{a_1r} + \overline{a_0} = \overline{0}$$

$$\overline{a_n}(\overline{r})^n + \overline{a_{n-1}}(\overline{r})^{n-1} + \cdots + \overline{a_2}(\overline{r})^2 + \overline{a_1r} + \overline{a_0} = \overline{0}$$

Theorem 4.6.C (continued)

Theorem 4.6.C. Conjugate Pairs Theorem. Let f be a polynomial function whose coefficients are real numbers. If $r = a + bi$ is a zero of f, then the complex conjugate $\overline{r} = a - bi$ is also a zero of f.

Proof (continued). ...

$$\overline{a_n}(\overline{r})^n + \overline{a_{n-1}}(\overline{r})^{n-1} + \cdots + \overline{a_2}(\overline{r})^2 + \overline{a_1}\overline{r} + \overline{a_0} = \overline{0}$$

$$a_n(\overline{r})^n + a_{n-1}(\overline{r})^{n-1} + \cdots + a_2(\overline{r})^2 + a_1\overline{r} + a_0 = 0$$

where the last equation follows since the conjugate of a real number is the number itself (Theorem A.4.A(a)). The last equation shows that $f(\overline{r}) = 0$ so that $\overline{r}$ is also a zero of f, as claimed. □

Corollary 4.6.D

Corollary 4.6.D. A polynomial function f of odd degree with real coefficients has at least one real zero.

Proof. Because complex zeros occur as conjugate pairs for a polynomial function with real coefficients by the Conjugate Pairs Theorem, Theorem 4.6.C, then there will always be an even number of zeros that are not real numbers. Consequently, since f is of odd degree, one of its zeros has to be a real number. □

Page 240 Number 16

Page 240 Number 16. Suppose polynomial function f with real coefficients is of degree 6 and has zeros i, $3 - 2i$, $-2 + i$. Find the remaining zeros of f.

Solution. Since f has real coefficients, then by the Conjugate Pairs Theorem, Theorem 4.6.C, for each complex zero r of f the complex number $\overline{r}$ is a zero of f. Since $r_1 = i$, $r_2 = 3 - 2i$, and $r_3 = -2 + i$ are zeros of f, then $\overline{r}_1 = \overline{i} = -i$, $\overline{r}_2 = \overline{3 - 2i} = 3 + 2i$, and $\overline{r}_3 = \overline{-2 + i} = -2 - i$ are also zeros. So these 6 numbers must be the totality of the zeros of f by the Fundamental Theorem of Algebra, Theorem 4.6.A, since f is of degree 6. Hence, the remaining zeros of f are $\boxed{-i, 3 + 2i, -2 - i}$. □

Page 240 Number 22

Page 240 Number 22. Find a polynomial function f with real coefficients having degree 5 and zeros 1 of multiplicity 3, and $1 + i$. Answers will vary depending on the choice of leading coefficient.

Solution. Since f has real coefficients, then by the Conjugate Pairs Theorem, Theorem 4.6.C, for each complex zero r of f the complex number $\overline{r}$ is a zero of f. Since $r = 1 + i$ is a zero of f, then $\overline{r} = \overline{1 + i} = 1 - i$ is also a zero. Since f is degree 5 and we have 5 zeros (counting the zero 1 by its multiplicity 3) then by the Fundamental Theorem of Algebra, Theorem 4.6.A, these are all the zeros of f. By the Factor Theorem, Theorem 4.5.C, each zero corresponds to a factor so that f has the factors $(x - 1)$, $(x - 1)$, $(x - 1)$, $(x - (1 + i))$, and $(x - (1 - i))$.

Page 240 Number 22 (continued)

Page 240 Number 22. Find a polynomial function f with real coefficients having degree 5 and zeros 1 of multiplicity 3, and $1+i$. Answers will vary depending on the choice of leading coefficient.

Solution (continued). So $(x-1)^3(x-(1+i))(x-(1-i)) = (x-1)^3(x^2-2x+2)$ is a factor of f and by the Factor Theorem, there can be no other factors of f (since it has no other zeros). So we can take $\boxed{f(x) = (x-1)^3(x-(1+i))(x-(1-i)) = (x-1)^3(x^2-2x+2)}$. □

NOTE: We can also take any nonzero multiple of $(x-1)^3(x-(1+i))(x-(1-i)) = (x-1)^3(x^2-2x+2)$ as a possible answer.

Theorem 4.5.G

Theorem 4.5.G. Every polynomial function (with real coefficients) can be uniquely factored into a product of linear factors and/or irreducible quadratic factors.

Proof. Every complex polynomial function f of degree n has exactly n zeros by the Fundamental Theorem of Algebra, Theorem 4.6.A, and can therefore be factored into a product of n linear factors by the Factor Theorem, Theorem 4.5.C. If its coefficients are real, those zeros that are complex numbers will always occur as conjugate pairs by the Conjugate Pairs Theorem, Theorem 4.6.C. As a result, if $r = a + bi$ is a complex zero (so that $b \neq 0$), then so is $\overline{r} = \overline{a+bi} = a - bi$. Consequently, when the linear factors $x - r$ and $x - \overline{r}$ of f are multiplied together, we have get

$$(x-r)(x-\overline{r}) = x^2 - (r+\overline{r})x + r\overline{r} = x^2 - (2a)x + (a^2+b^2),$$

since $r + \overline{r} = (a+bi) + (a-bi) = 2a$ and $r\overline{r} = (a+bi)(a-bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2$.

Theorem 4.5.G (continued)

Theorem 4.5.G. Every polynomial function (with real coefficients) can be uniquely factored into a product of linear factors and/or irreducible quadratic factors.

Proof (continued). This second-degree polynomial has real coefficients $x^2 - (2a)x + (a^2+b^2)$ and is irreducible (over the real numbers) by the Factor Theorem since it has no real zeros (because the discriminant is $(-2a)^2 - 4(1)(a^2+b^2) = 4a^2 - 4a^2 - 4b^2 = -4b^2 < 0$). So, the factors of f are either linear factors (which are associated with the real zeros) or irreducible quadratic factors (which are associated with conjugate pairs of complex zeros). □

Page 241 Number 38

Page 241 Number 38. Find the complex zeros of polynomial function $f(x) = x^4 + 3x^3 - 19x^2 + 27x - 252$. Write f in factored form.

Proof. By Descartes' Rule of Signs, Theorem 4.5.E, the coefficients of $f(x)$ are (in order as given by standard form) $1, 3, -19, 27, -252$ and there are 3 sign changes so that f has either 3 or 1 positive real zeros. Now $f(-x) = x^4 - 3x^3 - 19x^2 - 27x - 252$ has coefficients $1, -3, -10, -27, -252$ and there is 1 sign change so that f has 1 negative real zero by Descartes' Rule of Signs. Notice that f is of degree 4 and has at least two real zeros. Now we look for real, rational zeros of f. The factors of the leading coefficient $a_4 = 1$ are ± 1, and the factors of the constant term $a_0 = -252$ are $\pm 1, \pm 2, \pm 3 \pm 4, \pm 6, \pm 7, \pm 9, \pm 12, \pm 14 \pm 18, \pm 21, \pm 28, \pm 36, \pm 42, \pm 63, \pm 84, \pm 126, \pm 252$ (wow!). By the Rational Zeros Theorem, Theorem 4.5.F, the divisors of $a_0 = -252$ are the potential rational zeros of f.

Page 241 Number 38 (continued 1)

Proof (continued). Omitting several of the computations, we find that $f(4) = (4)^4 + 3(4)^3 - 19(4)^2 + 27(4) - 252 = 0$, $f(-7) = (-7)^4 + 3(-7)^3 - 19(-7)^2 + 27(-7) - 252 = 0$ and none of the other potential rational zeros are in fact zeros. So by the Factor Theorem, Theorem 4.5.C, we know that $x - 4$ and $x - (-7) = x + 7$ are factors of f. So $(x - 4)(x + 7) = x^2 + 3x - 28$ is a factor of f. We perform long division with this divisor:

$$\begin{array}{r|rrrrrrrrr} & & & & & x^2 & & & + & 9 \\ \hline x^2 + 3x - 28 & x^4 & + & 3x^3 & - & 19x^2 & + & 27x & - & 252 \\ & x^4 & + & 3x^3 & - & 28x^2 & & & & \\ \hline & & & & & 9x^2 & + & 27x & - & 252 \\ & & & & & 9x^2 & + & 27x & - & 252 \\ \hline & & & & & & & & & 0 \end{array}$$

So we have $f(x) = (x^2 + 3x - 28)(x^2 + 9) = (x - 4)(x + 7)(x^2 + 9)$ and notice that $x^2 + 9$ is an irreducible quadratic (over the real numbers).

Page 241 Number 38 (continued 2)

Page 241 Number 38. Find the complex zeros of polynomial function $f(x) = x^4 + 3x^3 - 19x^2 + 27x - 252$. Write f in factored form.

Proof (continued). With $x^2 + 9 = 0$ we have $x = 3i$ or $x = -3i$ are zeros, by the Factor Theorem $x^2 + 9 = (x - 3i)(x + 3i)$. So the zeros of f are $\boxed{x = 4,\ x = -7,\ x = 3i, \text{ and } x = -3i}$. We can then factor f (notice that the leading coefficient is 1) as

$\boxed{f(x) = x^4 + 3x^3 - 19x^2 + 27x - 252 = (x - 4)(x + 7)(x - 3i)(x + 3i)}$. □