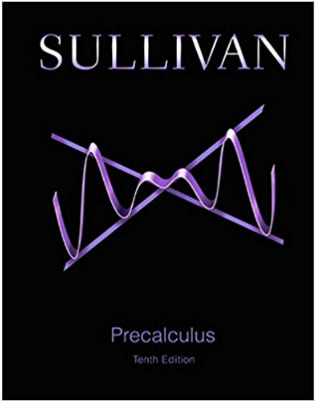
Precalculus 1 (Algebra)

Chapter 5. Exponential and Logarithmic Functions

5.1. Composite Functions—Exercises, Examples, Proofs



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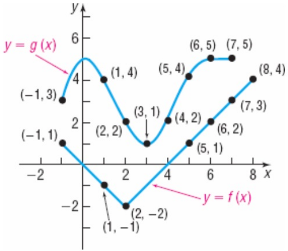
Page 254 Number 12

Page 254 Number 12. Evaluate each expression using the graphs of $y = f(x)$ and $y = g(x)$ shown here. **(a)** $(g \circ f)(1)$, **(b)** $(g \circ f)(5)$, **(c)** $(f \circ g)(0)$, and **(d)** $(f \circ g)(2)$.



Solution. **(a)** To find $(g \circ f)(1) = g(f(1))$, we first find $f(1)$ from the graph. The graph of f contains the point $(1, -1)$, so $f(1) = -1$. Hence $(g \circ f)(1) = g(f(1)) = g(-1)$. The graph of g contains the point $(-1, 3)$ so $g(-1) = 3$ and therefore $\boxed{(g \circ f)(1) = g(f(1)) = g(-1) = 3}$.

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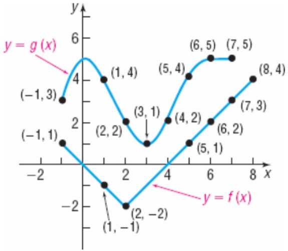
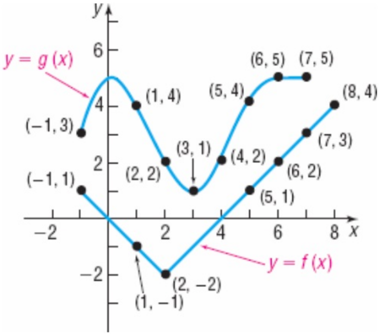


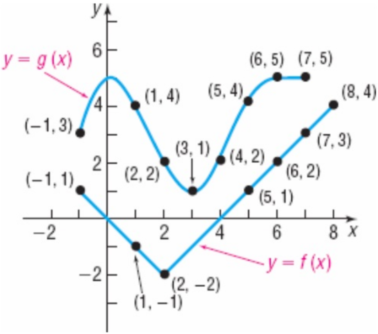
Solution. **(a)** To find $(g \circ f)(1) = g(f(1))$, we first find $f(1)$ from the graph. The graph of f contains the point $(1, -1)$, so $f(1) = -1$. Hence $(g \circ f)(1) = g(f(1)) = g(-1)$. The graph of g contains the point $(-1, 3)$ so $g(-1) = 3$ and therefore $\boxed{(g \circ f)(1) = g(f(1)) = g(-1) = 3}$.

Page 254 Number 12 (continued 1)



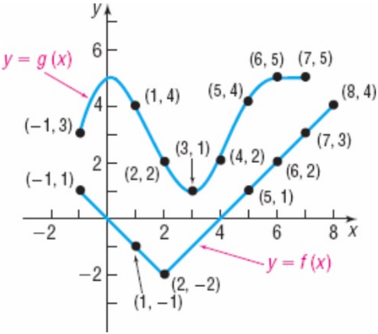
Solution (continued). (b) To find $(g \circ f)(5) = g(f(5))$, we first find $f(5)$ from the graph. The graph of f contains the point $(5, 1)$, so $f(5) = 1$. Hence $(g \circ f)(5) = g(f(5)) = g(1)$. The graph of g contains the point $(1, 4)$ so $g(1) = 4$ and therefore $\boxed{(g \circ f)(5) = g(f(5)) = g(1) = 4}$. □

Page 254 Number 12 (continued 2)



Solution (continued). (c) To find $(f \circ g)(0) = f(g(0))$, we first find $g(0)$ from the graph. The graph of g contains the point $(0, 5)$, so $g(0) = 5$. Hence $(f \circ g)(0) = f(g(0)) = f(5)$. The graph of f contains the point $(5, 1)$ so $f(5) = 1$ and therefore $\boxed{(f \circ g)(0) = f(g(0)) = f(5) = 1}$. □

Page 254 Number 12 (continued 3)



Solution (continued). (d) To find $(f \circ g)(2) = f(g(2))$, we first find $g(2)$ from the graph. The graph of g contains the point $(2, 2)$, so $g(2) = 2$. Hence $(f \circ g)(2) = f(g(2)) = f(2)$. The graph of f contains the point $(2, -2)$ so $f(2) = -2$ and therefore

$$\boxed{(f \circ g)(2) = f(g(2)) = f(2) = -2}.$$

□

Page 254 Number 22

Page 254 Number 22. Consider $f(x) = x^{3/2}$ and $g(x) = \dfrac{2}{x+1}$. Find **(a)** $(f \circ g)(4)$, **(b)** $(g \circ f)(2)$, **(c)** $(f \circ f)(1)$, **(d)** $(g \circ g)(0)$.

Proof. Notice that $f(g(x)) = f\left(\dfrac{2}{x+1}\right) = \left(\dfrac{2}{x+1}\right)^{3/2}$ and $g(f(x)) = g(x^{3/2}) = \dfrac{2}{x^{3/2}+1}$.

(a) We have $(f \circ g)(4) = f(g(4)) = \left(\dfrac{2}{(4)+1}\right)^{3/2} = \boxed{(2/5)^{3/2}}$. □

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(b) We have $(g \circ f)(2) = g(f(2)) = \boxed{\dfrac{2}{2^{3/2}+1}}$. □

(c) We have $(f \circ f)(1) = f(f(1)) = f((1)^{3/2}) = f(1) = 1^{3/2} = \boxed{1}$. □

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(d) We have $(g \circ g)(0) = g(g(0)) = g\left(\frac{2}{(0)+1}\right) = g(2) = \frac{2}{(2)+1} = \boxed{2/3}$. □

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$(g \circ g)(0) = g(g(0)) = g\left(\frac{2}{(0)+1}\right) = g(2) = \frac{2}{(2)+1} = \boxed{2/3}$. □

Page 255 Number 30

Page 255 Number 30. State the domain of the composite functions $f \circ g$ and $g \circ f$ where $f(x) = 1/(x+3)$ and $g(x) = -2/x$.

Solution. We compose the functions to find

$$(f \circ g)(x) = f(g(x)) = f(-2/x) = \frac{1}{(-2/x)+3} = \frac{1}{(-2/x)+3}\left(\frac{x}{x}\right)$$

$$= \frac{x}{-2(x/x)+3x} = \frac{x}{-2+3x} \text{ if } x \neq 0,$$

so the domain of $f \circ g$ is all real numbers except 0 and $2/3$; that is, the $\boxed{\text{domain of } f \circ g \text{ is } (-\infty, 0) \cup (0, 2/3) \cup (2/3, \infty)}$.

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Solution. Similarly we compose:

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x+3}\right) = \frac{-2}{1/(x+3)} = \frac{-2}{1/(x+3)}\left(\frac{x+3}{x+3}\right)$$

$$= \frac{-2(x+3)}{(x+3)/(x+3)} = -2x - 6 \text{ if } x \neq -3.$$

So the domain of $g \circ f$ is all real numbers except -3; that is, the $\boxed{\text{domain of } g \circ f \text{ is } (-\infty, -3) \cup (-3, \infty)}$. □

Page 255 Number 36

Page 255 Number 36. State the domain of the composite functions $f \circ g$ and $g \circ f$ where $f(x) = x^2 + 4$ and $g(x) = \sqrt{x-2}$.

Proof. We compose the functions to find

$$(f \circ g)(x) = f(\sqrt{x-2}) = (\sqrt{x-2})^2 + 4 = (x-2) + 4 \text{ if } x \geq 2$$

$$= x + 2 \text{ if } x \geq 2.$$

So the domain of $f \circ g$ is all real x with $x \geq 2$; that is, the $\boxed{\text{domain of } f \circ g \text{ is } [2, \infty)}$.

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Similarly we compose:

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 4) = \sqrt{(x^2 + 4) - 2} = \sqrt{x^2 + 2}.$$

Since $x^2 + 2 \geq 0$ for all real x, the domain of $g \circ f$ is all real numbers $\mathbb{R}$. □

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Since $x^2 + 2 \geq 0$ for all real x, the $\boxed{\text{domain of } g \circ f \text{ is all real numbers } \mathbb{R}}$. □

Page 255 Number 44

Page 255 Number 44. Show that $(f \circ g)(x) = (g \circ f)(x) = x$ for $f(x) = 4 - 3x$ and $g(x) = (4 - x)/3$.

Proof. Composing, we have

$$(f \circ g)(x) = f(g(x)) = f((4 - x)/3)$$

$$= 4 - 3((4 - x)/3) = 4 - (4 - x) = 4 - 4 + x = x,$$

as claimed.

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Also,

$$(g \circ f)(x) = g(f(x)) = g(4 - 3x)$$

$$= (4 - (4 - 3x))/3 = (4 - 4 + 3x)/3 = 3x/3 = x,$$

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$$= (4 - (4 - 3x))/3 = (4 - 4 + 3x)/3 = 3x/3 = x,$$

as claimed. □

Page 255 Number 48

Page 255 Number 48. Find functions f and g so that $(f \circ g)(x) = H(x) = (1 + x^2)^3$.

Proof. For $H(x) = (1 + x^2)^3$, we want the "outer" function to be the cubing function and the "inner" function to be $1 + x^2$ (the inner function is the stuff that is cubed in H).

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Proof. For $H(x) = (1 + x^2)^3$, we want the "outer" function to be the cubing function and the "inner" function to be $1 + x^2$ (the inner function is the stuff that is cubed in H). So we set $g(x) = 1 + x^2$ and $f(x) = x^3$. Composing we have

$$(f \circ g)(x) = f(g(x)) = f(1 + x^2) = (1 + x^2)^3 = H(x),$$

as desired. So $\boxed{f(x) = x^3 \text{ and } g(x) = 1 + x^2}$. □

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$$(f \circ g)(x) = f(g(x)) = f(1 + x^2) = (1 + x^2)^3 = H(x),$$

as desired. So $\boxed{f(x) = x^3 \text{ and } g(x) = 1 + x^2}$. □

Page 255 Number 60

Page 255 Number 60. The volume V (in cubic meters) of a hot-air balloon of radius r (in meters) is given by $V(r) = \frac{4}{3}\pi r^3$. If the radius r as a function of time t (in seconds) is $r(t) = \frac{2}{3}t^3$, where $t \geq 0$, then find the volume V as a function of the time t.

Proof. Since V is a function of r, $V(r)$, and r is a function of t, $r(t)$, to get V as a function of t we take the composition $(V \circ r)(t) = V(r(t))$.

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Proof. Since V is a function of r, $V(r)$, and r is a function of t, $r(t)$, to get V as a function of t we take the composition $(V \circ r)(t) = V(r(t))$. We then have

$$(V \circ r)(t) = V(r(t)) = V\left(\frac{2}{3}t^3\right) = \frac{4}{3}\pi\left(\frac{2}{3}t^3\right)^3$$

$$= \frac{4}{3}\pi\frac{2^3}{3^3}(t^3)^3 = \frac{(4)(8)}{(3)(27)}\pi t^9 = \frac{32}{81}\pi t^9.$$

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Proof. Since V is a function of r, $V(r)$, and r is a function of t, $r(t)$, to get V as a function of t we take the composition $(V \circ r)(t) = V(r(t))$. We then have

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So volume V as a function of time t is $\boxed{V = \frac{32}{81}\pi t^9}$. □

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So volume V as a function of time t is $\boxed{V = \frac{32}{81}\pi t^9}$. □