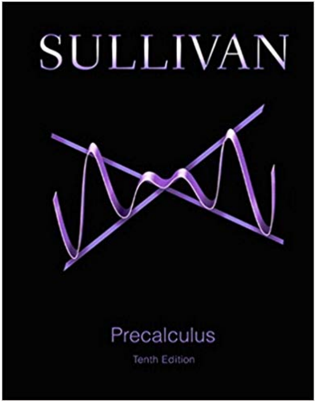
Precalculus 1 (Algebra)

Chapter 5. Exponential and Logarithmic Functions

5.6. Logarithmic and Exponential Equations—Exercises, Examples, Proofs



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Page 311 Number 10

Page 311 Number 10. Solve $\log_5(2x+3) = \log_5 3$.

Solution. By Theorem 5.5.A(1), $a^{\log_a M} = M$, so to eliminate the logarithms base 5 we make both sides exponents of 5 (that is, we take the equal sides as inputs into the base 5 exponential function 5^x): $5^{\log_5(2x+3)} = 5^{\log_5 3}$, or $2x+3=3$, or $2x=0$, or $\boxed{x=0}$. Notice that $x=0$ is in fact in the domain of both log functions. □

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Page 311 Number 18

Page 311 Number 18. Solve: $\log x + \log(x - 21) = 2$.

Solution. By Theorem 5.5.A(3), $\log x + \log(x - 21) = \log(x(x - 21)) = \log(x^2 - 21x)$. So we need $\log(x^2 - 21x) = 2$ and to eliminate the common logarithm function (i.e., the base 10 logarithm) we make both sides exponents of 10 to get: $10^{\log(x^2-21x)} = 10^2$, or $x^2 - 21x = 100$, or $x^2 - 21x - 100 = 0$, or $(x - 25)(x + 4) = 0$, or $x = -4$ and $x = 25$.

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Page 311 Number 40

Page 311 Number 40. Solve $\ln x - 3\sqrt{\ln x} + 2 = 0$.

Solution. Here, we introduce variable $u = \sqrt{\ln x}$ so that $\ln x - 3\sqrt{\ln x} + 2 = 0$ becomes the quadratic equation $u^2 - 3u + 2 = 0$ or $(u-1)(u-2) = 0$. So we need $u = 1$ or $u = 2$.

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Page 311 Number 44

Page 311 Number 44. Solve $3^x = 14$.

Solution. We take a natural logarithm of both sides to get $\ln 3^x = \ln 14$ or, by Theorem 5.5.A(5), $x \ln 3 = \ln 14$, or $\boxed{x = \frac{\ln 14}{\ln 3} \approx 2.402}$. □

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Page 312 Number 60

Page 312 Number 60. Solve $2^{2x} + 2^{x+2} - 12 = 0$.

Solution. We rewrite $2^{2x} + 2^{x+2} - 12 = 0$ as $(2^x)^2 + 2^2 2^x - 12 = 0$ or $(2^x)^2 + 4(2^x) - 12 = 0$. We introduce variable $u = 2^x$ so that the equation becomes $u^2 + 4u - 12 = 0$, or $(u+6)(u-2) = 0$, or $2^x = u = -6$ and $2^x = u = 2$.

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Page 312 Number 96

Page 312 Number 96. Let $f(x) = \log_3(x+5)$ and $g(x) = \log_3(x-1)$. **(a)** Solve $f(x) = 2$. What point is on the graph of f? **(b)** Solve $g(x) = 3$. What point is on the graph of g? **(c)** Solve $f(x) = g(x)$. Do the graphs of f and g intersect? If so, where? **(d)** Solve $(f+g)(x) = 3$. **(e)** $(f-g)(x) = 2$.

Solution. (a) To solve $f(x) = 2$, we consider $f(x) = \log_3(x+5) = 2$. Exponentiating base 3 gives $3^{\log_3(x+5)} = 3^2$ or, by Theorem 5.5.A(1), $x+5 = 9$, or $\boxed{x = 4}$. So the $\boxed{\text{point } (4, f(4)) = (4, 2) \text{ is on the graph of } f}$. □

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(b) To solve $g(x) = 3$, we consider $g(x) = \log_3(x-1) = 3$. Exponentiating base 3 gives $3^{\log_3(x-1)} = 3^3$, or $x - 1 = 27$, or $\boxed{x = 28}$. So the $\boxed{\text{point } (28, g(28)) = (28, 3) \text{ is on the graph of } g}$. □

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(e) $(f-g)(x) = 2$.

Solution. (a) To solve $f(x) = 2$, we consider $f(x) = \log_3(x+5) = 2$. Exponentiating base 3 gives $3^{\log_3(x+5)} = 3^2$ or, by Theorem 5.5.A(1), $x + 5 = 9$, or $\boxed{x = 4}$. So the $\boxed{\text{point } (4, f(4)) = (4, 2) \text{ is on the graph of } f}$. □

(b) To solve $g(x) = 3$, we consider $g(x) = \log_3(x-1) = 3$. Exponentiating base 3 gives $3^{\log_3(x-1)} = 3^3$, or $x - 1 = 27$, or $\boxed{x = 28}$. So the $\boxed{\text{point } (28, g(28)) = (28, 3) \text{ is on the graph of } g}$. □

Page 312 Number 96 (continued 1)

Page 312 Number 96. Let $f(x) = \log_3(x+5)$ and $g(x) = \log_3(x-1)$.
(c) Solve $f(x) = g(x)$. Do the graphs of f and g intersect? If so, where?
(d) Solve $(f+g)(x) = 3$. **(e)** $(f-g)(x) = 2$.

Solution (continued). (c) To solve $f(x) = g(x)$, we consider $f(x) = \log_3(x+5) = \log_3(x-1) = g(x)$. Exponentiating base 3 gives $3^{\log_3(x+5)} = 3^{\log_3(x-1)}$, or $x+5 = x-1$, or $0 = 6$; since this is certainly false then there is no such x and the equation has $\boxed{\text{no solution}}$. Since the function values of $f(x)$ and $g(x)$ are never equal for a given x value (that is, $(x, f(x)) \neq (x, g(x))$ for all x in the domains of f and g), then the graphs of f and g $\boxed{\text{do not intersect}}$. □

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(c) Solve $f(x) = g(x)$. Do the graphs of f and g intersect? If so, where?
(d) Solve $(f+g)(x) = 3$. **(e)** $(f-g)(x) = 2$.

Solution (continued). (c) To solve $f(x) = g(x)$, we consider $f(x) = \log_3(x+5) = \log_3(x-1) = g(x)$. Exponentiating base 3 gives $3^{\log_3(x+5)} = 3^{\log_3(x-1)}$, or $x+5 = x-1$, or $0 = 6$; since this is certainly false then there is no such x and the equation has $\boxed{\text{no solution}}$. Since the function values of $f(x)$ and $g(x)$ are never equal for a given x value (that is, $(x, f(x)) \neq (x, g(x))$ for all x in the domains of f and g), then the graphs of f and g $\boxed{\text{do not intersect}}$. □

Page 312 Number 96 (continued 2)

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Solution (continued). (d) To solve $(f+g)(x) = 3$ we observe that $(f+g)(x) = \log_3(x+5) + \log_3(x-1) = \log_3((x+5)(x-1))$ (by Theorem 5.5.A(3)), so we consider $\log_3((x+5)(x-1)) = 3$. Exponentiating base 3 gives $3^{\log_3((x+5)(x-1))} = 3^3$, or $(x+5)(x-1) = 27$, or $x^2 + 4x - 5 = 27$, or $x^2 + 4x - 32 = 0$, or $(x+8)(x-4) = 0$, or $x = -8$ and $x = 4$. But notice that $x = -8$ is not in the domain of either f or g, so the solution is $\boxed{x = 4}$. □

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(e) To solve $(f-g)(x) = 2$ we observe that $(f-g)(x) = \log_3(x+5) - \log_3(x-1) = \log_3\left(\frac{x+5}{x-1}\right)$ (by Theorem 5.5.A(4)), so we consider $\log_3((x+5)/(x-1)) = 2$. Exponentiating base 3 gives $3^{\log_3((x+5)/(x-1))} = 3^2$, or $(x+5)/(x-1) = 9$, or $x+5 = 9(x-1)$ where $x \neq 1$, or $x+5 = 9x-9$, or $8x = 14$, or $\boxed{x = 14/8 = 7/4}$. □

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Solution (continued). (d) To solve $(f+g)(x) = 3$ we observe that $(f+g)(x) = \log_3(x+5) + \log_3(x-1) = \log_3((x+5)(x-1))$ (by Theorem 5.5.A(3)), so we consider $\log_3((x+5)(x-1)) = 3$. Exponentiating base 3 gives $3^{\log_3((x+5)(x-1))} = 3^3$, or $(x+5)(x-1) = 27$, or $x^2 + 4x - 5 = 27$, or $x^2 + 4x - 32 = 0$, or $(x+8)(x-4) = 0$, or $x = -8$ and $x = 4$. But notice that $x = -8$ is not in the domain of either f or g, so the solution is $\boxed{x = 4}$. □

(e) To solve $(f-g)(x) = 2$ we observe that

$$(f-g)(x) = \log_3(x+5) - \log_3(x-1) = \log_3\left(\frac{x+5}{x-1}\right)$$ (by Theorem 5.5.A(4)), so we consider $\log_3((x+5)/(x-1)) = 2$. Exponentiating base 3 gives $3^{\log_3((x+5)/(x-1))} = 3^2$, or $(x+5)/(x-1) = 9$, or $x+5 = 9(x-1)$ where $x \neq 1$, or $x + 5 = 9x - 9$, or $8x = 14$, or $\boxed{x = 14/8 = 7/4}$. □

Page 313 Number 106

Page 313 Number 106. The population of the world in 2014 was 7.14 billion people and was growing at a rate of 1.1% per year. Assuming that this growth rate continues, the model $P(t) = 7.14(1.011)^{t-2014}$ represents the population P (in billions of people) in year t. **(a)** According to this model, when will the population of the world be 9 billion people? **(b)** According to this model, when will the population of the world be 12.5 billion people?

Solution. **(a)** We set $P(t) = 9$ and solve for t: $P(t) = 7.14(1.011)^{t-2014} = 9$. We take natural logarithms of both sides of this equation to get

$$\ln\left(7.14(1.011)^{t-2014}\right) = \ln 9, \text{ or } \ln 7.14 + \ln(1.011)^{t-2014} = \ln 9,$$

$$\text{or } \ln 7.14 + (t-2014)\ln 1.011 = \ln 9, \text{ or } (t-2014)\ln 1.011 = \ln 9 - \ln 7.14,$$

$$\text{or } t-2014 = (\ln 9 - \ln 7.14)/\ln 1.011, \text{ or } t = 2014 + (\ln 9 - \ln 7.14)/\ln 1.011.$$

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Solution. **(a)** We set $P(t) = 9$ and solve for t: $P(t) = 7.14(1.011)^{t-2014} = 9$. We take natural logarithms of both sides of this equation to get

$$\ln\left(7.14(1.011)^{t-2014}\right) = \ln 9, \text{ or } \ln 7.14 + \ln(1.011)^{t-2014} = \ln 9,$$

$$\text{or } \ln 7.14 + (t-2014)\ln 1.011 = \ln 9, \text{ or } (t-2014)\ln 1.011 = \ln 9 - \ln 7.14,$$

$$\text{or } t-2014 = (\ln 9 - \ln 7.14)/\ln 1.011, \text{ or } t = 2014 + (\ln 9 - \ln 7.14)/\ln 1.011.$$

Page 313 Number 106 (continued)

Solution (continued). So $t = 2014 + (\ln 9 - \ln 7.14)/\ln 1.011 \approx 2014 + 21.162 = 2035.162$, so the population will reach $\boxed{\text{9 billion in the year 2035}}$. □

(b) We set $P(t) = 12.5$ and solve for t: $P(t) = 7.14(1.011)^{t-2014} = 12.5$. We take natural logarithms of both sides of this equation to get

$$\ln\left(7.14(1.011)^{t-2014}\right) = \ln 12.5, \text{ or } \ln 7.14 + \ln(1.011)^{t-2014} = \ln 12.5,$$

$$\text{or } \ln 7.14 + (t-2014)\ln 1.011 = \ln 12.5,$$

$$\text{or } (t-2014)\ln 1.011 = \ln 12.5 - \ln 7.14,$$

$$\text{or } t - 2014 = (\ln 12.5 - \ln 7.14)/\ln 1.011,$$

$$\text{or } t = 2014 + (\ln 12.5 - \ln 7.14)/\ln 1.011.$$

So $t = 2014 + (\ln 12.5 - \ln 7.14)/\ln 1.011 \approx 2014 + 51.190 = 2065.190$, so the population will reach $\boxed{\text{12.5 billion in the year 2065}}$. □

Page 313 Number 106 (continued)

Solution (continued). So $t = 2014 + (\ln 9 - \ln 7.14)/\ln 1.011 \approx 2014 + 21.162 = 2035.162$, so the population will reach $\boxed{\text{9 billion in the year 2035}}$. □

(b) We set $P(t) = 12.5$ and solve for t: $P(t) = 7.14(1.011)^{t-2014} = 12.5$. We take natural logarithms of both sides of this equation to get

$$\ln\left(7.14(1.011)^{t-2014}\right) = \ln 12.5, \text{ or } \ln 7.14 + \ln(1.011)^{t-2014} = \ln 12.5,$$

$$\text{or } \ln 7.14 + (t - 2014)\ln 1.011 = \ln 12.5,$$

$$\text{or } (t - 2014)\ln 1.011 = \ln 12.5 - \ln 7.14,$$

$$\text{or } t - 2014 = (\ln 12.5 - \ln 7.14)/\ln 1.011,$$

$$\text{or } t = 2014 + (\ln 12.5 - \ln 7.14)/\ln 1.011.$$

So $t = 2014 + (\ln 12.5 - \ln 7.14)/\ln 1.011 \approx 2014 + 51.190 = 2065.190$, so the population will reach $\boxed{\text{12.5 billion in the year 2065}}$. □