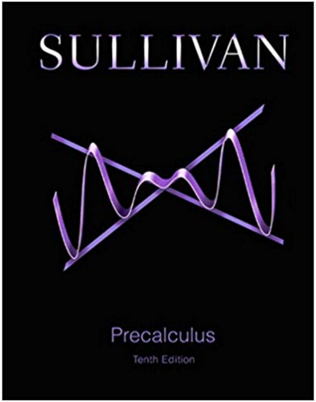
Precalculus 1 (Algebra)

Appendix A. Review

A.10. *n*th Roots; Rational Exponents—Exercises, Examples, Proofs



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Page A88 Numbers 12, 14, 22, 30, 36, and 38

Page A88 Numbers 12, 14, 22, 30, 36, and 38. Simplify each expression: **(12)** $\sqrt[4]{16}$, **(14)** $\sqrt[3]{-1}$, **(22)** $\sqrt[3]{\dfrac{3xy^2}{81x^4y^2}}$, **(30)** $\left(\sqrt[3]{3}\sqrt{10}\right)^4$, **(36)** $2\sqrt{12} - 3\sqrt{27}$, and **(38)** $\left(\sqrt{5} - 2\right)\left(\sqrt{5} + 3\right)$.

Solution. (12) For $\sqrt[4]{16}$, we seek a nonnegative real number a that when raised to the 4th power gives 16, $a^4 = 16$. We have $a = 2$ since $2^4 = 2 \times 2 \times 2 \times 2 = 16$, and so $\boxed{\sqrt[4]{16} = 2}$.

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(12) For $\sqrt[3]{-1}$, we seek a real number a that when raised to the 3rd power gives -1, $a^3 = -1$. We have $a = -1$ since $(-1)^3 = (-1) \times (-1) \times (-1) = -1$, and so $\boxed{\sqrt[3]{-1} = -1}$.

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Page A88 Numbers 12, 14, 22, 30, 36, and 38 (continued 1)

Solution. (22) We first simplify under the radical and have, by properties of exponents, $\sqrt[3]{\dfrac{3xy^2}{81x^4y^2}} = \sqrt[3]{\dfrac{3}{81}\dfrac{x}{x^4}\dfrac{y^2}{y^2}} = \sqrt[3]{\dfrac{1}{27}\dfrac{1}{x^3}(1)}$ if $y \neq 0$. Now by properties of roots (Notes A.10.A), $\sqrt[3]{\dfrac{1}{27}\dfrac{1}{x^3}(1)} = \sqrt[3]{\dfrac{1}{27}}\sqrt[3]{\dfrac{1}{x^3}}$ and since $(1/3)^3 = 1/27$ and $(1/x)^3 = 1/x^3$ then $\sqrt[3]{\dfrac{1}{27}}\sqrt[3]{\dfrac{1}{x^3}} = \dfrac{1}{3}\dfrac{1}{x} = \dfrac{1}{3x}$. So $\sqrt[3]{\dfrac{3xy^2}{81x^4y^2}} = \boxed{\dfrac{1}{3x} \text{ if } y \neq 0}$. □

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$\left(\sqrt[3]{3}\sqrt{10}\right)^4 = (\sqrt[3]{3})^4(\sqrt{10})^4 = (\sqrt[3]{3})^3\sqrt[3]{3}(\sqrt{10})^2(\sqrt{10})^2.$

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$\left(\sqrt[3]{3}\sqrt{10}\right)^4 = (\sqrt[3]{3})^4(\sqrt{10})^4 = (\sqrt[3]{3})^3\sqrt[3]{3}(\sqrt{10})^2(\sqrt{10})^2$. Now $(\sqrt[3]{3})^3 = 3$ and $(\sqrt{10})^2 = 10$, so
$\left(\sqrt[3]{3}\sqrt{10}\right)^4 = (\sqrt[3]{3})^3\sqrt[3]{3}(\sqrt{10})^2(\sqrt{10})^2 = (3)\sqrt[3]{3}(10)(10) = \boxed{300\sqrt[3]{3}}$. □

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Solution. (36) We express the numbers under the radicals as products to get $2\sqrt{12} - 3\sqrt{27} = 2\sqrt{3 \times 2^2} - 3\sqrt{3^3}$ so that we can apply properties of roots (Notes A.10.A) to get
$2\sqrt{12} - 3\sqrt{27} = 2\sqrt{2^2 \times 3} - 3\sqrt{3^3} = 2\sqrt{2^2}\sqrt{3} - 3\sqrt{3^2}\sqrt{3}$. Since $\sqrt{2^2} = 2$ and $\sqrt{3^2} = 3$, then
$2\sqrt{12} - 3\sqrt{27} = 2(2\sqrt{3}) - 3(3\sqrt{3}) = (4 - 9)\sqrt{3} = \boxed{-5\sqrt{3}}$. □

(38) By FOIL we have
$\left(\sqrt{5} - 2\right)\left(\sqrt{5} + 3\right) = (\sqrt{5})(\sqrt{5}) + (\sqrt{5})(3) + (-2)(\sqrt{5}) + (-2)(3) = \sqrt{5}^2 + (3 - 2)\sqrt{5} - 6 = 5 + \sqrt{5} - 6 = \boxed{-1 + \sqrt{5}}$. □

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Page A88 numbers 52 and 58

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(52) $\dfrac{-\sqrt{3}}{\sqrt{8}}$ and **(58)** $\dfrac{-3}{\sqrt{5}+4}$.

Solution. (52) To get a rational denominator, we multiply by a version of 1 that eliminates the radical term in the denominator:

$$\frac{-\sqrt{3}}{\sqrt{8}} = \frac{-\sqrt{3}}{\sqrt{8}}(1) = \frac{-\sqrt{3}}{\sqrt{8}}\left(\frac{\sqrt{8}}{\sqrt{8}}\right) = \frac{-\sqrt{3}\sqrt{8}}{\sqrt{8}\sqrt{8}} = \frac{-\sqrt{3}\sqrt{2\times 2^2}}{(\sqrt{8})^2}$$

$$= \frac{-\sqrt{3}\sqrt{2}\sqrt{2^2}}{8} = \frac{-2\sqrt{2}\sqrt{3}}{8} = \frac{-\sqrt{2}\sqrt{3}}{4} = \boxed{\frac{-\sqrt{6}}{4}}.$$

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Page A88 numbers 52 and 58 (continued)

(58) $\dfrac{-3}{\sqrt{5}+4}$.

Solution. **(58)** To get a rational denominator, we multiply by a version of 1 that eliminates the radical term in the denominator. We do so by taking advantage of the identity $a^2 - b^2 = (a-b)(a+b)$:

$$\frac{-3}{\sqrt{5}+4} = \frac{-3}{\sqrt{5}+4}(1) = \frac{-3}{\sqrt{5}+4}\left(\frac{\sqrt{5}-4}{\sqrt{5}-4}\right)$$

$$= \frac{-3(\sqrt{5}-4)}{(\sqrt{5}+4)(\sqrt{5}-4)} = \frac{-3(\sqrt{5}-4)}{(\sqrt{5})^2-(4)^2} = \frac{-3\sqrt{5}+12}{(5)-16}$$

$$= \frac{-3\sqrt{5}+12}{-11} = \frac{-(3\sqrt{5}-12)}{-(11)} = \boxed{\frac{3\sqrt{5}-12}{11}}.$$

□

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Page A88 Number 66. Solve the equation $\sqrt{12 - x} = x$.

Solution. First, notice that x is a square root and so $x \geq 0$. We square both sides to eliminate the square root, so $(\sqrt{12 - x})^2 = x^2$ or $12 - x = x^2$ or $x^2 + x - 12 = 0$ or (factoring) $(x - 3)(x + 4) = 0$.

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Note. When squaring both sides of an equation, we loose negative signs and run the risk of introducing *extraneous roots* (see Note A.6.B). Here, the value $x = -4$ is an extraneous root and, in fact, not a solution to the original equation.

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Page A88 Numbers 80 and 88

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Solution. (80) Since a negative exponent indicates a reciprocal, we have $-25^{-1/2} = \dfrac{-1}{25^{1/2}}$. Since an exponent of $1/2$ indicates a square root, then we have $-25^{-1/2} = \dfrac{-1}{\sqrt{25}} = \dfrac{-1}{\sqrt{5^2}} = \boxed{\dfrac{-1}{5}}$. □

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(88) By the properties of roots, we have $\dfrac{(xy)^{1/4}(x^2y^2)^{1/2}}{(x^2y)^{3/4}} = \dfrac{x^{1/4}y^{1/4}(x^2)^{1/2}(y^2)^{1/2}}{(x^2)^{3/4}y^{3/4}} = \dfrac{x^{1/4}y^{1/4}x^1y^1}{x^{3/2}y^{3/4}} = x^{1/4}x^1x^{-3/2}y^{1/4}y^1y^{-3/4} = x^{1/4+1-3/2}y^{1/4+1-3/4} = x^{-1/4}y^{1/2} = \boxed{\dfrac{y^{1/2}}{x^{1/4}}}$. □

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$x^{1/4+1-3/2}y^{1/4+1-3/4} = x^{-1/4}y^{1/2} = \boxed{\dfrac{y^{1/2}}{x^{1/4}}}$. □

Page A89 Number 112

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Solution. We can factor out the lowest powers of $(6x+1)$ and $(4x-3)$, which are $(6x+1)^{1/3}$ and $(4x-3)^{1/2}$:

$$\begin{aligned} & 6(6x+1)^{1/3}(4x-3)^{3/2}+6(6x+1)^{4/3}(4x-3)^{1/2} \\ &= 6(6x+1)^{1/3}(4x-3)^{1/2}\left((4x-3)^{3/2-1/2}+(6x+1)^{4/3-1/3}\right) \\ &= 6(6x+1)^{1/3}(4x-3)^{1/2}\left((4x-3)+(6x+1)\right) \\ &= \boxed{6(6x+1)^{1/3}(4x-3)^{1/2}(10x-2)}. \end{aligned}$$

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$$6(6x+1)^{1/3}(4x-3)^{3/2}+6(6x+1)^{4/3}(4x-3)^{1/2}$$

$$= 6(6x+1)^{1/3}(4x-3)^{1/2}\left((4x-3)^{3/2-1/2}+(6x+1)^{4/3-1/3}\right)$$

$$= 6(6x+1)^{1/3}(4x-3)^{1/2}((4x-3)+(6x+1))$$

$$= \boxed{6(6x+1)^{1/3}(4x-3)^{1/2}(10x-2)}.$$

□