Precalculus 1 (Algebra)

Appendix A. Review

A.2. Geometry Essentials—Exercises, Examples, Proofs

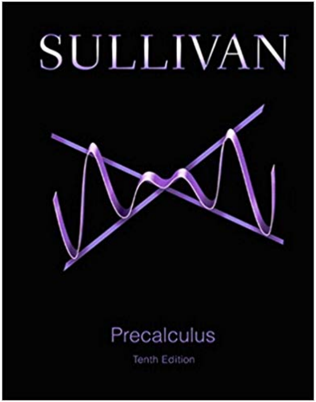


Table of contents

Page A19 Number 14

Page A19 Number 14. If the lengths of the legs of a right triangle are $a = 6$ and $b = 8$, then find the length of the hypotenuse c.

Solution. By the Pythagorean Theorem, Theorem A.2.A, with the symbols introduced we have $a^2 + b^2 = c^2$. So c satisfies $c^2 = (6)^2 + (8)^2 = 36 + 64 = 100$. With $c^2 = 100$, we have either $c = -10$ or $c = 10$. Since c is a length, then it cannot be negative so we must have the length of the hypotenuse as $\boxed{c = 10}$. □

Page A19 Number 14

Page A19 Number 14. If the lengths of the legs of a right triangle are $a = 6$ and $b = 8$, then find the length of the hypotenuse c.

Solution. By the Pythagorean Theorem, Theorem A.2.A, with the symbols introduced we have $a^2 + b^2 = c^2$. So c satisfies $c^2 = (6)^2 + (8)^2 = 36 + 64 = 100$. With $c^2 = 100$, we have either $c = -10$ or $c = 10$. Since c is a length, then it cannot be negative so we must have the length of the hypotenuse as $\boxed{c = 10}$. □

Page A19 Number 26

Page A19 Number 26. The lengths of the sides of a triangle are 4, 5, and 7. Determine if the triangle is a right triangles. If it is, identify the hypotenuse.

Solution. The Converse of the Pythagorean Theorem, Theorem A.2.B, states that if the length of one side squared is the sum of the squares of the other two lengths then we have a right triangle. But $4^2 = 16$, $5^2 = 25$, and $7^2 = 49$. Since $16 \neq 25 + 49 = 74$, $25 \neq 16 + 49 = 65$, and $49 \neq 16 + 25 = 41$, then the Converse of the Pythagorean Theorem does not apply.

Page A19 Number 26

Page A19 Number 26. The lengths of the sides of a triangle are 4, 5, and 7. Determine if the triangle is a right triangles. If it is, identify the hypotenuse.

Solution. The Converse of the Pythagorean Theorem, Theorem A.2.B, states that if the length of one side squared is the sum of the squares of the other two lengths then we have a right triangle. But $4^2 = 16$, $5^2 = 25$, and $7^2 = 49$. Since $16 \neq 25 + 49 = 74$, $25 \neq 16 + 49 = 65$, and $49 \neq 16 + 25 = 41$, then the Converse of the Pythagorean Theorem does not apply.

By the Pythagorean Theorem, Theorem A.2.A, *if* we have a right triangle then the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two legs. Since this cannot hold, then the $\boxed{\text{triangle with sides of given lengths cannot be a right triangle}}$. (Technically, we are using the *contrapositive* of the Pythagorean Theorem). □

Page A19 Number 26

Page A19 Number 26. The lengths of the sides of a triangle are 4, 5, and 7. Determine if the triangle is a right triangles. If it is, identify the hypotenuse.

Solution. The Converse of the Pythagorean Theorem, Theorem A.2.B, states that if the length of one side squared is the sum of the squares of the other two lengths then we have a right triangle. But $4^2 = 16$, $5^2 = 25$, and $7^2 = 49$. Since $16 \neq 25 + 49 = 74$, $25 \neq 16 + 49 = 65$, and $49 \neq 16 + 25 = 41$, then the Converse of the Pythagorean Theorem does not apply.

By the Pythagorean Theorem, Theorem A.2.A, *if* we have a right triangle then the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two legs. Since this cannot hold, then the $\boxed{\text{triangle with sides of given lengths cannot be a right triangle}}$. (Technically, we are using the *contrapositive* of the Pythagorean Theorem). □

page A21 Number 56

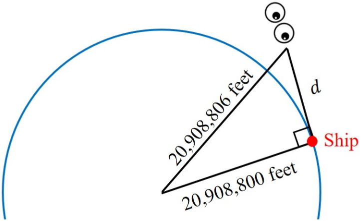
page A21 number 56. A person who is 6 feet tall is standing on the beach in Fort Lauderdale, Florida, and looks out onto the Atlantic Ocean. Suddenly a ship appears on the horizon. How far is the ship from the shore? HINT: The radius of the Earth is 3960 miles and 1 mile equals 5280 feet.

Solution. The Earth is a sphere, so we cut this sphere with a plane passing through the person, the ship, and the center of the Earth. This gives the following cross section (certainly not to scale!):

page A21 Number 56

page A21 number 56. A person who is 6 feet tall is standing on the beach in Fort Lauderdale, Florida, and looks out onto the Atlantic Ocean. Suddenly a ship appears on the horizon. How far is the ship from the shore? HINT: The radius of the Earth is 3960 miles and 1 mile equals 5280 feet.

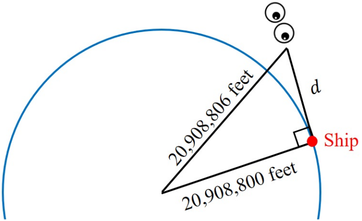
Solution. The Earth is a sphere, so we cut this sphere with a plane passing through the person, the ship, and the center of the Earth. This gives the following cross section (certainly not to scale!):



page A21 Number 56

page A21 number 56. A person who is 6 feet tall is standing on the beach in Fort Lauderdale, Florida, and looks out onto the Atlantic Ocean. Suddenly a ship appears on the horizon. How far is the ship from the shore? HINT: The radius of the Earth is 3960 miles and 1 mile equals 5280 feet.

Solution. The Earth is a sphere, so we cut this sphere with a plane passing through the person, the ship, and the center of the Earth. This gives the following cross section (certainly not to scale!):



page A21 Number 56 (continued)

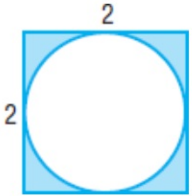
Solution (continued). The radius of the Earth in feet is (3960 miles)(5280 feet/mile) $=$ (3960)(5280) feet $=$ 20,908,800 feet. So the distance from the center of the Earth to the eyes of the observer (technically, to the top of the observer's head...) is 20,908,806 feet. We draw a line tangent to the circle passing through the observer's eyes. Lines tangent to a circle are perpendicular to a radius of the circle containing the point of tangency, so we get the pictured right triangle and we want to find d. Since we have a right triangle, then the Pythagorean Theorem gives $(20{,}908{,}800 \text{ feet})^2 + d^2 = (20{,}908{,}806 \text{ feet})^2$ or $d^2 = (20{,}908{,}806)^2 - (20{,}908{,}800)^2 \text{ feet}^2 = 205{,}905{,}636$ feet. Since d is a distance it is positive and so $d = \sqrt{205{,}905{,}636}$ feet $\approx 15{,}840$ feet. Now $(15{,}840 \text{ feet})(1/5280 \text{miles/feet}) = 3$ miles. So the ship is $\boxed{\text{3 miles away}}$. □

page A21 Number 56 (continued)

Solution (continued). The radius of the Earth in feet is (3960 miles)(5280 feet/mile) $=$ (3960)(5280) feet $=$ 20,908,800 feet. So the distance from the center of the Earth to the eyes of the observer (technically, to the top of the observer's head...) is 20,908,806 feet. We draw a line tangent to the circle passing through the observer's eyes. Lines tangent to a circle are perpendicular to a radius of the circle containing the point of tangency, so we get the pictured right triangle and we want to find d. Since we have a right triangle, then the Pythagorean Theorem gives $(20{,}908{,}800 \text{ feet})^2 + d^2 = (20{,}908{,}806 \text{ feet})^2$ or $d^2 = (20{,}908{,}806)^2 - (20{,}908{,}800)^2 \text{ feet}^2 = 205{,}905{,}636$ feet. Since d is a distance it is positive and so $d = \sqrt{205{,}905{,}636}$ feet $\approx 15{,}840$ feet. Now (15,840 feet)(1/5280miles/feet) $=$ 3 miles. So the ship is $\boxed{\text{3 miles away}}$. □

Page A20 Number 40

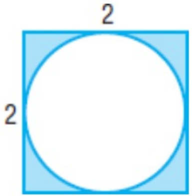
Page A20 Number 40. Find the shaded area.



Solution. We apply the formulae of Note A.2.A. The square can be viewed as a rectangle of length $\ell = 2$ and width $w = 2$, so the area of the square is $A_s = \ell w = (2)(2) = 4$. The circle has a diameter $d = 2$ and a radius $r = 1$, so it's area is $A_c = \pi r^2 = \pi(1)^2 = \pi$. The shaded area is the area of the square minus the area of the circle and so is $\boxed{A = A_s - A_c = 4 - \pi}$. □

Page A20 Number 40

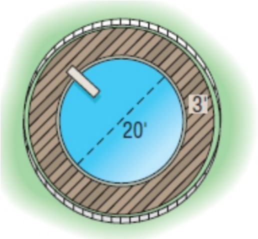
Page A20 Number 40. Find the shaded area.



Solution. We apply the formulae of Note A.2.A. The square can be viewed as a rectangle of length $\ell = 2$ and width $w = 2$, so the area of the square is $A_s = \ell w = (2)(2) = 4$. The circle has a diameter $d = 2$ and a radius $r = 1$, so it's area is $A_c = \pi r^2 = \pi(1)^2 = \pi$. The shaded area is the area of the square minus the area of the circle and so is $\boxed{A = A_s - A_c = 4 - \pi}$. □

Page A20 Number 52

Page A20 Number 52. A circular swimming pool that is 20 feet in diameter is enclosed by a wooden deck that is 3 feet wide. What is the area of the deck? How much fence is required to enclose the deck?

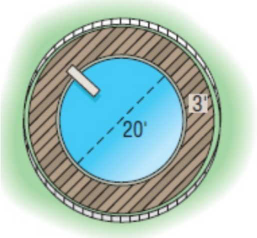


Solution. The pool has a diameter of $d_p = 20$ feet, or a radius of $r_p = 10$ feet. The diameter of the circle determined by the outer fence is $d_f = 26$ feet, which gives a corresponding radius of $r_f = 13$ feet. So the area of the deck, by Note A.2.A, is

$A = (\pi r_f^2 - \pi r_p^2)$ feet$^2 = (\pi(13)^2 - \pi(10)^2)$ feet$^2 = \boxed{69\pi \text{ feet}^2}$.

Page A20 Number 52

Page A20 Number 52. A circular swimming pool that is 20 feet in diameter is enclosed by a wooden deck that is 3 feet wide. What is the area of the deck? How much fence is required to enclose the deck?

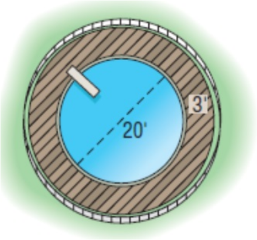


Solution. The pool has a diameter of $d_p = 20$ feet, or a radius of $r_p = 10$ feet. The diameter of the circle determined by the outer fence is $d_f = 26$ feet, which gives a corresponding radius of $r_f = 13$ feet. So the area of the deck, by Note A.2.A, is

$A = (\pi r_f^2 - \pi r_p^2)$ feet$^2 = (\pi(13)^2 - \pi(10)^2)$ feet$^2 = \boxed{69\pi \text{ feet}^2}$.

Page A20 Number 52 (continued)

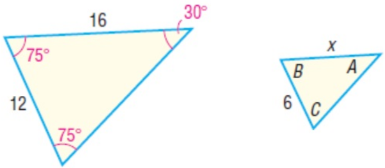
Page A20 Number 52. A circular swimming pool that is 20 feet in diameter is enclosed by a wooden deck that is 3 feet wide. What is the area of the deck? How much fence is required to enclose the deck?



Solution (continued). The amount of fencing is equal to the circumference of a circle of radius $r_f = 13$ feet, and so (by Note A.2.A) $\boxed{C = 2\pi r_f = 2\pi(13 \text{ feet}) = 26\pi \text{ feet of fencing is needed}}$. □

Page A20 Number 44

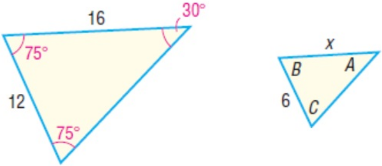
Page A20 Number 44. The given triangles are similar. Find the missing length x and the missing angles A, B, and C.



Solution. Since, in similar triangles, corresponding angles are equal then we must have $\boxed{A = 30^\circ \text{ and } B = C = 75^\circ}$.

Page A20 Number 44

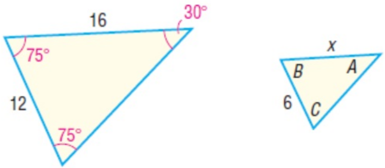
Page A20 Number 44. The given triangles are similar. Find the missing length x and the missing angles A, B, and C.



Solution. Since, in similar triangles, corresponding angles are equal then we must have $\boxed{A = 30^\circ \text{ and } B = C = 75^\circ}$. In similar triangles, the lengths of the corresponding sides are proportional. The side of length 12 in the triangle on the left corresponds to the side of length 6 in the triangle on the right, so the proportion of the lengths of the sides of the triangle on the left to the length of the sides of the triangle on the right is $2:1$. The side of length 16 in the triangle on the left corresponds to side AB in the triangle on the right, and so $\boxed{x = 8}$. □

Page A20 Number 44

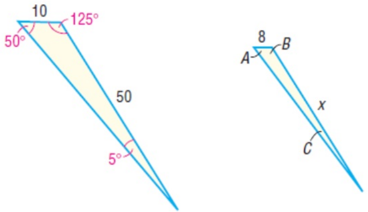
Page A20 Number 44. The given triangles are similar. Find the missing length x and the missing angles A, B, and C.



Solution. Since, in similar triangles, corresponding angles are equal then we must have $\boxed{A = 30^\circ \text{ and } B = C = 75^\circ}$. In similar triangles, the lengths of the corresponding sides are proportional. The side of length 12 in the triangle on the left corresponds to the side of length 6 in the triangle on the right, so the proportion of the lengths of the sides of the triangle on the left to the length of the sides of the triangle on the right is $2 : 1$. The side of length 16 in the triangle on the left corresponds to side AB in the triangle on the right, and so $\boxed{x = 8}$. □

Page A20 Number 46

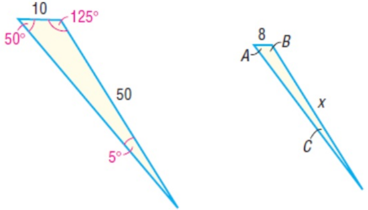
Page A20 Number 46. The given triangles are similar. Find the missing length x and the missing angles A, B, and C.



Solution. Since, in similar triangles, corresponding angles are equal then we must have $\boxed{A = 50^\circ,\ B = 125^\circ, \text{ and } C = 5^\circ}$.

Page A20 Number 46

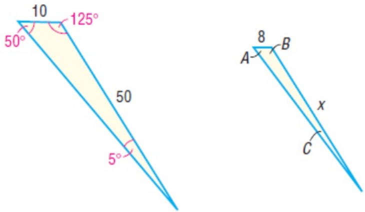
Page A20 Number 46. The given triangles are similar. Find the missing length x and the missing angles A, B, and C.



Solution. Since, in similar triangles, corresponding angles are equal then we must have $\boxed{A = 50^\circ,\ B = 125^\circ, \text{ and } C = 5^\circ}$. In similar triangles, the lengths of the corresponding sides are proportional. The side of length 10 in the triangle on the left corresponds to the side of length 8 in the triangle on the right, so the proportion of the lengths of the sides of the triangle on the left to the length of the sides of the triangle on the right is $10 : 8$ or $5 : 4$. The side of length 50 in the triangle on the left corresponds to side BC in the triangle on the right, and so $\boxed{x = 40}$. □

Page A20 Number 46

Page A20 Number 46. The given triangles are similar. Find the missing length x and the missing angles A, B, and C.



Solution. Since, in similar triangles, corresponding angles are equal then we must have $\boxed{A = 50^\circ,\ B = 125^\circ, \text{ and } C = 5^\circ}$. In similar triangles, the lengths of the corresponding sides are proportional. The side of length 10 in the triangle on the left corresponds to the side of length 8 in the triangle on the right, so the proportion of the lengths of the sides of the triangle on the left to the length of the sides of the triangle on the right is $10 : 8$ or $5 : 4$. The side of length 50 in the triangle on the left corresponds to side BC in the triangle on the right, and so $\boxed{x = 40}$. □