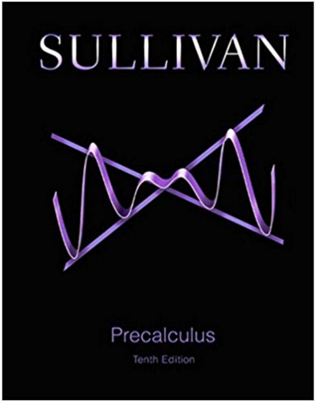
Precalculus 1 (Algebra)

Appendix A. Review

A.3. Polynomials—Exercises, Examples, Proofs



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Page A30 Number 24

Page A30 Number 24. Tell whether the expression $3x^2 + 4$ is a monomial. If it is, name the variable and the coefficient and give the degree of the monomial. If it is not a monomial, state why not.

Solution. The expression $3x^2 + 4$ is $\boxed{\text{not a monomial}}$, but is a binomial which is the sum of the two monomials $3x^2$ and 4. Notice that monomial $3x^2$ is degree 2 with coefficient 3 and variable x. □

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Page A30 Numbers 48 and 50

Page A30 numbers 48 and 50. Multiply, as indicated and express your answer as a single polynomial in standard form: **(48)** $(3x+1)(2x+1)$, and **(50)** $(x-1)(x+1)$.

Solution. (48) Applying FOIL we have
$(3x+1)(2x+1) = (3x)(2x) + (3x)(1) + (1)(2x) + (1)(1) =$
$6x^2 + 3x + 2x + 1 = \boxed{6x^2 + 5x + 1}$. □

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$(x)(x) + (x)(1) + (-1)(x) + (-1)(1) = x^2 + x - x - 1 = \boxed{x^2 - 1}$. □

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Page A30 Numbers 66 and 74

Page A30 Numbers 66 and 74. Find the quotient and the remainder. Check your work by verifying that (Quotient)(Divisor) + remainder = Dividend. **(66)** Divide $5x^4 - x^2 + x - 2$ by $x^2 + 2$, and **(74)** divide $1 - x^2 + x^4$ by $x^2 - x + 1$.

Solution. (66) We perform long division:

$$
\begin{array}{r|rrrrrrr}
 & & & 5x^2 & & & - & 11 \\
\hline
x^2+2 & 5x^4 & - & x^2 & + & x & - & 2 \\
 & 5x^4 & + & 10x^2 & & & & \\
\hline
 & & - & 11x^2 & + & x & - & 2 \\
 & & - & 11x^2 & & & - & 22 \\
\hline
 & & & & & x & + & 20
\end{array}
$$

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x^2+2 & 5x^4 & - & x^2 & + & x & - & 2 \\
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 & & - & 11x^2 & + & x & - & 2 \\
 & & - & 11x^2 & & & - & 22 \\
\hline
 & & & & & x & + & 20
\end{array}$$

So the quotient is $5x^2 - 11$ and the remainder is $x + 20$.

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Solution. (66) We perform long division:

$$\begin{array}{r|rrrrrrr} & 5x^2 & & & & -11 \\ \hline x^2+2 & 5x^4 & - & x^2 & + & x & - & 2 \\ & 5x^4 & + & 10x^2 & & & & \\ \hline & & - & 11x^2 & + & x & - & 2 \\ & & - & 11x^2 & & & - & 22 \\ \hline & & & & & x & + & 20 \end{array}$$

So $\boxed{\text{the quotient is } 5x^2 - 11 \text{ and the remainder is } x + 20}$. To check: $(5x^2 - 11)(x^2 + 2) + (x + 20) = (5x^2)(x^2) + (5x^2)(2) + (-11)(x^2) + (-11)(2) + x + 20 = 5x^4 + 10x^2 - 11x^2 - 22 + x + 20 = 5x^4 - x^2 + x - 2$. □

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So $\boxed{\text{the quotient is } 5x^2 - 11 \text{ and the remainder is } x + 20}$. To check: $(5x^2 - 11)(x^2 + 2) + (x + 20) = (5x^2)(x^2) + (5x^2)(2) + (-11)(x^2) + (-11)(2) + x + 20 = 5x^4 + 10x^2 - 11x^2 - 22 + x + 20 = 5x^4 - x^2 + x - 2$. □

Page A30 Numbers 66 and 74 (continued)

Solution (continued). (74) We perform long division:

$$
\begin{array}{r|rrrrrrrrr}
 & & & & & x^2 & + & x & - & 1 \\
\hline
x^2 - x + 1 & x^4 & & & - & x^2 & & & + & 1 \\
 & x^4 & - & x^3 & + & x^2 & & & & \\
\hline
 & & & x^3 & - & 2x^2 & & & & \\
 & & & x^3 & - & x^2 & + & x & & \\
\hline
 & & & & - & x^2 & - & x & + & 1 \\
 & & & & - & x^2 & + & x & - & 1 \\
\hline
 & & & & & & - & 2x & + & 2
\end{array}
$$

So $\boxed{\text{the quotient is } x^2 + x - 1 \text{ and the remainder is } -2x + 2}$.

Page A30 Numbers 66 and 74 (continued)

Solution (continued). (74) We perform long division:

$$\begin{array}{r|l} & x^2 + x - 1 \\ \hline x^2 - x + 1 & x^4 - x^2 + 1 \\ & x^4 - x^3 + x^2 \\ \hline & x^3 - 2x^2 \\ & x^3 - x^2 + x \\ \hline & -x^2 - x + 1 \\ & -x^2 + x - 1 \\ \hline & -2x + 2 \end{array}$$

So $\boxed{\text{the quotient is } x^2 + x - 1 \text{ and the remainder is } -2x + 2}$. To check: $(x^2+x-1)(x^2-x+1)+(-2x+2) = x^2(x^2-x+1)+x(x^2-x+1)-1(x^2-x+1)-2x+2 = x^4-x^3+x^2+x^3-x^2+x-x^2+x-1-2x+2 = x^4-x^2+1$. □

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Solution (continued). (74) We perform long division:

$$
\begin{array}{r|rrrrrrrrr}
 & & & & & x^2 & + & x & - & 1 \\ \hline
x^2 - x + 1 & x^4 & & & - & x^2 & & & + & 1 \\
 & x^4 & - & x^3 & + & x^2 & & & & \\ \hline
 & & & x^3 & - & 2x^2 & & & & \\
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So $\boxed{\text{the quotient is } x^2 + x - 1 \text{ and the remainder is } -2x + 2}$. To check: $(x^2+x-1)(x^2-x+1)+(-2x+2) = x^2(x^2-x+1)+x(x^2-x+1)-1(x^2-x+1)-2x+2 = x^4-x^3+x^2+x^3-x^2+x-x^2+x-1-2x+2 = x^4-x^2+1$. □

Page A30 Number 78

Page A30 Number 78. Factor completely $x^2 - 9$. If it cannot be factored, say it is prime.

Solution. The special product "Difference of Two Square" states $(x-a)(x+a) = x^2 - a^2$. The right hand side of this equation with $a = 3$ is $x^2 - 9$, so we can factor $x^2 - 9$ as $\boxed{(x-3)(x+3)}$. □

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Page A30 Number 83

Page A30 Number 83. Factor completely $x^2 - 10x + 21$. If it cannot be factored, say it is prime.

Solution. We look for integers a and b such that $a + b = B = -10$ and $ab = C = 21$. From the condition $ab = 21$ we see that we must use ± 1 and ± 21, or ± 3 and ± 7. Since $(-3) + (-7) = -10$, we take $a = -3$ and $b = -7$ to get $x + a = x - 3$ and $x + b = x - 7$. Then $\boxed{x^2 - 10x + 21 = (x - 3)(x - 7)}$. □

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Page A30 Number 124

Page A30 Number 124. Factor completely $x^4 + x^3 + x + 1$. If it cannot be factored, say it is prime.

Solution. We first factor by grouping:
$x^4+x^3+x+1 = (x^4+x^3)+(x+1) = x^3(x+1)+1(x+1) = (x^3+1)(x+1)$.

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Solution. We first factor by grouping:
$x^4+x^3+x+1 = (x^4+x^3)+(x+1) = x^3(x+1)+1(x+1) = (x^3+1)(x+1)$.
Since $x^3 + 1 = x^3 + 1^3$ is a Sum of Two Cubes, we have by one of the "special products" that $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$ and so (with $a = 1$) we know $x^3 + 1 = (x + 1)(x^2 - x + 1)$.

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$x^4 + x^3 + x + 1 = (x^3 + 1)(x + 1) = (x + 1)(x^2 - x + 1)(x + 1) =$
$\boxed{(x^2 - x + 1)(x + 1)^2}$.

Notice that $x^2 - x + 1$ cannot be factored over the integers since this would require integers a and b with $ab = 1$ and $a + b = -1$, which cannot happen. □

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Notice that $x^2 - x + 1$ cannot be factored over the integers since this would require integers a and b with $ab = 1$ and $a + b = -1$, which cannot happen. □

Page A30 Number 98

Page A30 Number 98. Factor completely $8x^2 + 6x - 2$. If it cannot be factored, say it is prime.

Solution. First, we factor out the common factor 2 to get $2(4x^2 + 3x - 1)$.

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Solution. First, we factor out the common factor 2 to get $2(4x^2 + 3x - 1)$. Now in $Ax^2 + Bx + C = 4x^2 + 3x - 1$, we have $AC = (4)(-1) = -4$. Now we need integers a and b such that $ab = AC = -4$ and $a + b = B = 3$. So we can take $a = 4$ and $b = -1$.

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Solution. First, we factor out the common factor 2 to get $2(4x^2 + 3x - 1)$. Now in $Ax^2 + Bx + C = 4x^2 + 3x - 1$, we have $AC = (4)(-1) = -4$. Now we need integers a and b such that $ab = AC = -4$ and $a + b = B = 3$. So we can take $a = 4$ and $b = -1$. This gives $Ax^2 + Bx + C = Ax^2 + ax + bx + C = 4x^2 + 4x - x - 1 = 4x(x+1) - (x+1) = (4x-1)(x+1)$. Hence $\boxed{8x^2 + 6x - 2x = 2(4x-1)(x+1)}$. □

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Page A30 Number 100

Page A30 Number 100. Factor completely $x^4 - 1$. If it cannot be factored, say it is prime.

Solution. The special product "Difference of Two Square" states $(x - a)(x + a) = x^2 - a^2$. We have $x^4 - 1 = (x^2)^2 - 1^2$ so we factor to get $(x^2 - 1)(x^2 + 1)$.

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Page A31 Number 126

Page A31 Number 126. Determine the number that should be added to complete the square of the expression $p^2 + 14p$. Then factor the expression.

Solution. We know $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$. So with $b = 14$ we add $(b/2)^2 = (14/2)^2 = 7^2 = 49$ to the given expression to get $\boxed{p^2 + 14p + 49 = (p + (14/2))^2 = (p + 7)^2}$. □

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