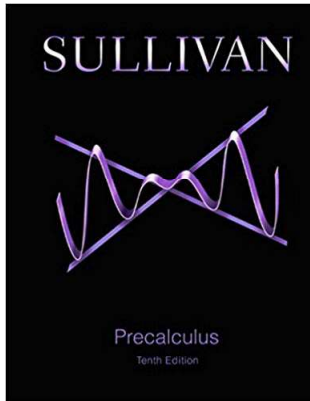


Page A51 Number 16

Precalculus 1 (Algebra)

Appendix A. Review

A.6. Solving Equations—Exercises, Examples, Proofs



Page A51 Number 16. Solve $3x = -24$.

Solution. Dividing both sides by 3 gives the equivalent equation $(3x)/3 = (-24)/3$ or $x = -8$. The solution set is $\{-8\}$. \square

Page A51 Number 24

Page A51 Number 24. Solve $3 - 2x = 2 - x$.

Solution. Adding $2x$ to both sides (to get the x 's on one side) gives the equivalent equation $(3 - 2x) + 2x = (2 - x) + 2x$ or $3 = 2 + x$. Subtracting 2 from both sides (to isolate x) of this gives the equivalent equation $(3) - 2 = (2 + x) - 2$ or $1 = x$. So $x = 1$ and the solution set is $\{1\}$. \square

Page A51 Number 28

Page A51 Number 28. Solve $5 - (2x - 1) = 10$.

Solution. First, we simplify the left hand side to get $5 - (2x - 1) = 5 - 2x + 1 = 6 - 2x$. So an equivalent equation is $6 - 2x = 10$. Adding $2x$ to both sides of this equation (to get the x 's on one side) gives the equivalent equation $(6 - 2x) + 2x = 10 + 2x$ or $6 = 10 + 2x$. Subtracting 10 from both sides (to start isolating the x 's) gives the equivalent equation $(6) - 10 = (10 + 2x) - 10$ or $-4 = 2x$. To isolate x , we divide both sides by 2 to get the equivalent equation $(-4)/2 = (2x)/2$ or $-2 = x$. So $x = -2$ or the solution set is $\{-2\}$. \square

Page A51 Number 36

Page A51 Number 36. Solve $(x + 2)(x - 3) = (x - 3)^2$.

Solution. We try to use the Zero-Product Property, so we want 0 on one side. We subtract $(x + 2)(x - 3)$ from both sides to get the equivalent equation $(x + 2)(x - 3) - (x + 2)(x - 3) = (x - 3)^2 - (x + 2)(x - 3)$ or $0 = (x - 3)^2 - (x + 2)(x - 3)$. Factoring out the common factor $x - 3$ we get the equivalent equation $0 = (x - 3)((x - 3) - (x + 2))$ or $(x - 3)(-5) = 0$. Dividing both sides by -5 gives $(x - 3)(-5)/(-5) = 0/(-5)$ or $x - 3 = 0$. Adding 3 to both sides we have $x = 3$. So $x = 3$ and the solution set is $\{3\}$. \square

Page A51 Number 54

Page A51 Number 54. Solve $|1 - 2z| = 3$.

Solution. The equation $|1 - 2z| = 3$ is satisfied if either $1 - 2z = 3$ or $1 - 2z = -3$, so we find the solution set by solving these equations separately. For $1 - 2z = 3$, we have $-2 = 2z$ or $z = -1$. For $1 - 2z = -3$, we have $4 = 2z$ or $z = 2$. So we have either $z = -1$ or $z = 2$ and the solution set is $\{-1, 2\}$. \square

Page A51 Number 48

Page A51 Number 48. Solve $\frac{1}{2x + 3} + \frac{1}{x - 1} = \frac{1}{(2x + 3)(x - 1)}$.

Solution. First, we get a common denominator on the left hand side:

$$\frac{1}{2x + 3} + \frac{1}{x - 1} = \frac{1}{2x + 3} \left(\frac{x - 1}{x - 1} \right) + \frac{1}{x - 1} \left(\frac{2x + 3}{2x + 3} \right) =$$

$$\frac{x - 1}{(2x + 3)(x - 1)} + \frac{2x + 3}{(x - 1)(2x + 3)} = \frac{(x - 1) + (2x + 3)}{(x - 1)(2x + 3)} =$$

$$\frac{3x + 2}{(x - 1)(2x + 3)}.$$

So the original equation is equivalent to $\frac{3x + 2}{(x - 1)(2x + 3)} = \frac{1}{(2x + 3)(x - 1)}$. Since the denominators are the same, then the quotients can only be equal if the numerators are equal. So we need $3x + 2 = 1$ (speeding up the process a little) or $3x = -1$ or $x = -1/3$. So $x = -1/3$ and the solution set is $\{-1/3\}$. \square

Page A51 Number 74

Page A51 Number 74. Solve $x(x + 1) = 12$.

Solution. First we multiply the left side out to get $x(x + 1) = x^2 + x$, and then the given equation is equivalent to $x^2 + x = 12$ or $x^2 + x - 12 = 0$. We can factor to get the equivalent equation $x^2 + x - 12 = (x - 3)(x + 4) = 0$. So by the Zero-Product Property, either $x - 3 = 0$ or $x + 4 = 0$; that is, either $x = 3$ or $x = -4$. So the solution set is $\{-4, 3\}$. \square

Page A51 Number 86

Page A51 Number 86. Solve $(3x - 2)^2 = 4$ by taking square roots.

Solution. Taking square roots of both sides of the given equation gives $\sqrt{(3x - 2)^2} = \sqrt{4}$ or $|3x - 2| = 2$. (BEWARE that $\sqrt{x^2} = |x|$ by Note A.1.E., it does not simply equal x .) So the original equation is satisfied if either $3x - 2 = 2$ or $3x - 2 = -2$. Solving $3x - 2 = 2$ we get $3x = 4$ or $x = 4/3$. Solving $3x - 2 = -2$ we get $3x = 0$ or $x = 0$. So we have either $x = 0$ or $x = 4/3$ and the solution set is $\{0, 4/3\}$. \square

Page A52 Number 92

Page A52 Number 92. Solve $2x^2 - 3x - 1 = 0$ by completing the square.

Solution. First, we need to consider

$2x^2 - 3x - 1 = 2\left(x^2 - \frac{3}{2}x - \frac{1}{2}\right) = 0$. To complete the square on the expression in the parentheses, we recall that $x^2 - bx + (b/2)^2 = (x - b/2)^2$ and see that we have $b = 3/2$ or $b/2 = 3/4$ here, and so $(b/2)^2 = (3/4)^2 = 9/16$. So we have $x^2 - \frac{3}{2}x + \frac{9}{16} = \left(x - \frac{3}{4}\right)^2$. Then the original equation is equivalent to $2x^2 - 3x - 1 = 2\left(x^2 - \frac{3}{2}x - \frac{1}{2}\right) = 2\left(x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{9}{16} - \frac{1}{2}\right) = 2\left(\left(x - \frac{3}{4}\right)^2 - \frac{9}{16} - \frac{1}{2}\right) = 0$ or $2\left(\left(x - \frac{3}{4}\right)^2 - \frac{17}{16}\right) = 0$. Dividing by 2 gives the equivalent equation $\left(x - \frac{3}{4}\right)^2 - \frac{17}{16} = 0$ or $\left(x - \frac{3}{4}\right)^2 = \frac{17}{16}$.

Page A52 Number 92 (continued 1)

Solution (continued). Now we take square roots of both sides of

$\left(x - \frac{3}{4}\right)^2 = \frac{17}{16}$ to get $\sqrt{\left(x - \frac{3}{4}\right)^2} = \sqrt{\frac{17}{16}}$ or $\left|x - \frac{3}{4}\right| = \sqrt{\frac{17}{16}} = \frac{\sqrt{17}}{\sqrt{16}} = \frac{\sqrt{17}}{4}$. That is, either $x - \frac{3}{4} = \frac{\sqrt{17}}{4}$ or $x - \frac{3}{4} = -\frac{\sqrt{17}}{4}$. Solving $x - \frac{3}{4} = \frac{\sqrt{17}}{4}$ gives $x = \frac{3}{4} + \frac{\sqrt{17}}{4} = \frac{3 + \sqrt{17}}{4}$. Solving $x - \frac{3}{4} = -\frac{\sqrt{17}}{4}$ gives $x = \frac{3}{4} - \frac{\sqrt{17}}{4} = \frac{3 - \sqrt{17}}{4}$. So either $x = \frac{3 + \sqrt{17}}{4}$ or $x = \frac{3 - \sqrt{17}}{4}$.

Page A52 Number 92 (continued 2)

Page A52 Number 92. Solve $2x^2 - 3x - 1 = 0$ by completing the square.

Solution (continued). That is, $x = \frac{3 \pm \sqrt{17}}{4}$ and the solution set is

$$\left\{ \frac{3 - \sqrt{17}}{4}, \frac{3 + \sqrt{17}}{4} \right\}.$$

Note. We could have used the quadratic formula (with $a = 2$, $b = -3$, and $c = -1$) and got the same answer:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)} \\ = \frac{3 \pm \sqrt{9 + 8}}{4} = \frac{3 \pm \sqrt{17}}{4}.$$

Theorem A.6.A

Theorem A.6.A. The Quadratic Formula. Consider the quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$. If $b^2 - 4ac < 0$ then this equation has no real solution. If $b^2 - 4ac \geq 0$ then the real solution(s) of this equation is (are) given by the *quadratic formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity $b^2 - 4ac$ is called the *discriminant*.

Proof. We derive the quadratic formula by completing the square:

$$\begin{aligned} ax^2 + bx + c &= 0, \quad a \neq 0 \\ \frac{ax^2 + bx + c}{a} &= \frac{0}{a} \quad (\text{divide both sides by } a) \\ x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \quad (\text{simplify}) \end{aligned}$$

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Theorem A.6.A (continued 1)

Proof (continued).

$$\begin{aligned} x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\ x^2 + \frac{b}{a}x &= -\frac{c}{a} \quad (\text{subtract } c/a \text{ from both sides}) \\ x^2 + \frac{b}{a}x + \left(\frac{b/a}{2}\right)^2 &= -\frac{c}{a} + \left(\frac{b/a}{2}\right)^2 \quad (\text{complete the square on the left} \\ &\quad \text{and keep equation balanced}) \\ x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} &= -\frac{c}{a} + \frac{b^2}{4a^2} \quad (\text{simplify}) \\ \left(x + \frac{b}{2a}\right)^2 &= -\frac{c}{a} + \frac{b^2}{4a^2} \quad (\text{factor the perfect square}) \\ \left(x + \frac{b}{2a}\right)^2 &= -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \quad (\text{get common denominator}) \end{aligned}$$

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Theorem A.6.A (continued 2)

Proof (continued).

$$\begin{aligned} \left(x + \frac{b}{2a}\right)^2 &= -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \quad (\text{simplify}) \\ x + \frac{b}{2a} &= \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad (\text{Square Root Method}) \\ x &= -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad (\text{subtract } b/2a \text{ from both sides}) \\ x &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2|a|} \quad (\text{simplify}) \\ x &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad (\text{the } \pm \text{ takes care of any negative} \\ &\quad \text{signs absorbed by the absolute value}) \end{aligned}$$

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Theorem A.6.A (continued 3)

Theorem A.6.A. The Quadratic Formula. Consider the quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$. If $b^2 - 4ac < 0$ then this equation has no real solution. If $b^2 - 4ac \geq 0$ then the real solution(s) of this equation is (are) given by the *quadratic formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity $b^2 - 4ac$ is called the *discriminant*.

Proof (continued).

$$\begin{aligned} x &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (\text{simplify}) \end{aligned}$$

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Page A52 Number 98

Page A52 Number 98. Use the quadratic formula to find the real solutions, if any, for $2x^2 + 5x + 3 = 0$.

Solution. With $a = 2$, $b = 5$, and $c = 3$ we have from the quadratic formula

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(5) \pm \sqrt{(5)^2 - 4(2)(3)}}{2(2)} \\ &= \frac{-5 \pm \sqrt{25 - 24}}{4} = \frac{-5 \pm 1}{4}. \end{aligned}$$

So we have that

either $x = (-5 + 1)/4 = -1$ or $x = (-5 - 1)/4 = -6/4 = -3/2$ and the solution set is $\{-1, -3/2\}$. \square

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Page A52 Number 104

Page A52 Number 104. Use the quadratic formula to find the real solutions, if any, for $x^2 + \sqrt{2}x - 2 = 0$.

Solution. With $a = 1$, $b = \sqrt{2}$, and $c = -2$ we have from the quadratic formula

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(\sqrt{2}) \pm \sqrt{(\sqrt{2})^2 - 4(1)(-2)}}{2(1)} \\ &= \frac{-\sqrt{2} \pm \sqrt{2 + 8}}{2} = \frac{-\sqrt{2} \pm \sqrt{10}}{2}. \end{aligned}$$

So we have that either $x = \frac{-\sqrt{2} - \sqrt{10}}{2}$ or $\frac{-\sqrt{2} + \sqrt{10}}{2}$ and the

solution set is $\left\{ \frac{-\sqrt{2} - \sqrt{10}}{2}, \frac{-\sqrt{2} + \sqrt{10}}{2} \right\}$. \square

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Page A52 Number 110

Page A52 Number 110. Use the discriminant to determine whether $2x^2 - 3x - 4 = 0$ has two unequal real solutions, a repeated real solution, or no real solution without solving the equation.

Solution. With $a = 2$, $b = -3$, and $c = -4$, the discriminant is $b^2 - 4ac = (-3)^2 - 4(2)(-4) = 9 + 32 = 41$. Since $b^2 - 4ac > 0$ then by Note A.6.C, there are two unequal real solutions. \square

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