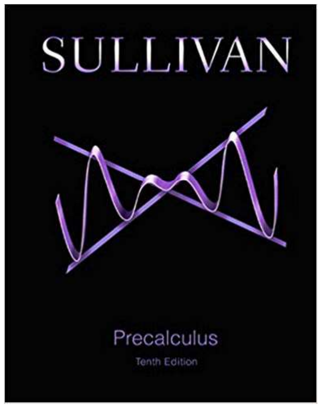
Precalculus 1 (Algebra)

Appendix A. Review

A.7. Complex Numbers; Quadratic Equations in the Complex Number System—Exercises, Examples, Proofs



Page A60 Numbers 12, 14 and 22

Page A60 Numbers 12, 14 and 22. Perform the indicated operation, and write each expression in the standard form $a + bi$.
(12) $(4+5i)+(-8+2i)$, **(14)** $(3-4i)-(-3-4i)$, and **(22)** $(5+3i)(2-i)$.

Solution. (12) By definition of complex number addition,
$(4+5i)+(-8+2i) = (4+(-8))+(5+2)i = \boxed{-4+7i}$. □

(14) By definition of complex number subtraction,
$(3-4i)-(-3-4i) = (3-(-3))+((-4)-(-4))i = \boxed{6}$. □

(22) By definition of complex number product,
$(5+3i)(2-i) = (5)(2)+(5)(-i)+(3i)(2)+(3i)(-i) =$
$10-5i+6i-3i^2 = 10-5i+6i-3(-1) = \boxed{13+i}$. □

Page A61 number 92

Page A61 number 92. Prove Theorem A.7.A(d): The conjugate of the product of two complex numbers equals the product of their conjugates: $\overline{zw} = \overline{z}\,\overline{w}$.

Proof. Let $z = a+bi$ and $w = c+di$ in standard form. Then by the definition of multiplication of complex numbers, $\overline{zw} = \overline{(a+bi)(c+di)} = \overline{(a)(c)+(a)(di)+(bi)(c)+(bi)(di)} = \overline{(ac+adi+bci+bdi^2)} = \overline{(ac-bd)+(ad+bc)i} = (ac-bd)-(ad+bc)i$.

Similarly, $\overline{z}\,\overline{w} = \overline{a+bi}\,\overline{c+di} = (a-bi)(c-di) = (a)(c)+(a)(-di)+(-bi)(c)+(-bi)(-di) = ac-adi-bci+bdi^2 = (ac-bd)-(ad+bc)i$.

Hence $\overline{zw} = \overline{z}\,\overline{w}$, since both equal $(ac-bd)-(ad+bc)i$. □

Page A60 Number 26

Page A60 Number 26. Perform the indicated operation, and write each expression in the standard form $a + bi$. **(26)** $\dfrac{13}{5-12i}$.

Solution. (26) We make the denominator real by multiplying by a version of 1 that involves the conjugate of the denominator:

$$\frac{13}{5-12i} = \frac{13}{5-12i}(1) = \frac{13}{5-12i}\left(\frac{5+12i}{5+12i}\right) = \frac{13(5+12i)}{(5-12i)(5+12i)}$$

$$= \frac{13(5+12i)}{5^2+12^2} = \frac{13(5+12i)}{169} = \frac{5+12i}{13} = \boxed{\frac{5}{13}+\frac{12}{13}i}.$$

□

Page A60 Number 30

Page A60 Number 30. Perform the indicated operation, and write each expression in the standard form $a + bi$. **(30)** $\dfrac{2+3i}{1-i}$.

Solution. (30) We make the denominator real by multiplying by a version of 1 that involves the conjugate of the denominator:

$$\frac{2+3i}{1-i} = \frac{2+3i}{1-i}(1) = \frac{2+3i}{1-i}\left(\frac{1+i}{1+i}\right) = \frac{(2+3i)(1+i)}{(1-i)(1+i)}$$

$$= \frac{(2+3i)(1+i)}{1^2+1^2} = \frac{(2+3i)(1+i)}{2} = \frac{(2)(1)+(2)(i)+(3i)(1)+(3i)(i)}{2}$$

$$= \frac{2+2i+3i+3(-1)}{2} = \frac{-1+5i}{2} = \boxed{\frac{-1}{2}+\frac{5}{2}i}.$$

□

Page A61 Numbers 36, 38, and 48

Page A61 Numbers 36, 38, and 48. Perform the indicated operation, and write each expression in the standard form $a + bi$. **(36)** i^{14}, **(38)** i^{-23}, and **(48)** $i^7 + i^5 + i^3 + i$.

Solution. (36) Since $i^4 = 1$, we have

$$i^{14} = (i^4)(i^4)(i^4)(i^2) = (1)(1)(1)(-1) = \boxed{-1}.$$

□

(38) Since $i^4 = 1$ then $i^{20} = (i^4)^5 = 1^5 = 1$, then $i^{23} = (i^{20})(i^3) = (1)(-i) = -i$. So

$$i^{-23} = \frac{1}{i^{23}} = \frac{1}{-i} = \frac{1}{-i}\left(\frac{i}{i}\right) = \frac{i}{-i^2} = \frac{i}{1} = \boxed{i}.$$

□

Page A61 Numbers 36, 38, and 48 (continued)

Page A61 Numbers 36, 38, and 48. Perform the indicated operation, and write each expression in the standard form $a + bi$. **(36)** i^{14}, **(38)** i^{-23}, and **(48)** $i^7 + i^5 + i^3 + i$.

Solution (continued. (48) Since $i^4 = 1$ then we can reduce the exponents of i by multiples of 4 to get

$$i^7 + i^5 + i^3 + i = (i^4)(i^3) + (i^4)(i) + i^3 + i$$

$$= i^3 + i + i^3 + i = 2i^3 + 2i = 2(-i) + 2i = \boxed{0}.$$

□

Page A61 Number 60

Page A61 Number 60. Solve in the complex number system $x^2 + 4x + 8 = 0$.

Solution. We use the quadratic formula (which involves the principal square root), $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ where $a = 1$, $b = 4$, and $c = 8$. So we have

$$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(8)}}{2(1)} = \frac{-4 \pm \sqrt{16-32}}{2}$$

$$= \frac{-4 \pm \sqrt{-16}}{2} = \frac{-4 \pm 4i}{2} = -2 \pm 2i.$$

So the solutions are $\boxed{x = -2-2i \text{ and } x = -2+2i}$, or the solution set is $\boxed{\{-2-2i, -2+2i\}}$. □

Page A61 Number 70

Page A61 Number 70. Solve in the complex number system $x^3 + 27 = 0$.

Solution. The expression $x^3 + 27 = x^3 + 3^3$ is a "Sum of Two Cubes," so from the formula $(x + a)(x^2 - ax + a^2) = x^3 + a^3$ (see A.3. Polynomials) we have $x^3 + 27 = (x + 3)(x^2 - 3x + 9)$. So we see that $x = -3$ is a solution. But we also have solutions when $x^2 - 3x + 9 = 0$, so we solve this with the quadratic formula to also get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)} = \frac{3 \pm \sqrt{9 - 36}}{2}$$

$$= \frac{3 \pm \sqrt{-27}}{2} = \frac{3 \pm \sqrt{9(-3)}}{2} = \frac{3 \pm 3\sqrt{-3}}{2} = \frac{3 \pm 3\sqrt{3}i}{2} = \frac{3}{2} \pm \frac{3\sqrt{3}}{2}i.$$

So the solutions are $\boxed{x = -3,\ x = 3/2 + (3\sqrt{3}/2)i, \text{ and } x = 3/2 - (3\sqrt{3}/2)i}$ and the solution set is $\boxed{\{-3, 3/2 + (3\sqrt{3}/2)i, 3/2 - (3\sqrt{3}/2)i\}}$. □

Page A61 Number 78

Page A61 Number 78. Without solving, determine the character of the solutions of $x^2 + 6 = 2x$ in the complex number system.

Solution. We rewrite the equation as $x^2 - 2x + 6 = 0$ and consider the discriminant $b^2 - 4ac$ where $a = 1$, $b = -2$, and $c = 6$. We have $b^2 - 4ac = (-2)^2 - 4(1)(6) = 4 - 24 = -20$. Since $b^2 - 4ac = -20 < 0$ then the equation has $\boxed{\text{two complex solutions}}$ that are not real (and are conjugates of each other). □