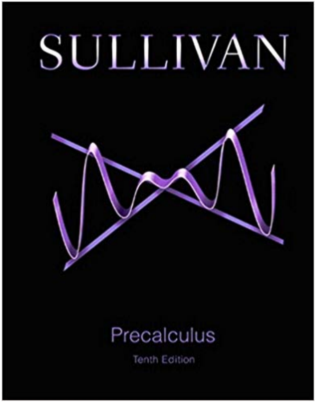
Precalculus 1 (Algebra)

Appendix A. Review

A.7. Complex Numbers; Quadratic Equations in the Complex Number System—Exercises, Examples, Proofs



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Page A60 Numbers 12, 14 and 22

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(12) $(4 + 5i) + (-8 + 2i)$, **(14)** $(3 - 4i) - (-3 - 4i)$, and **(22)** $(5 + 3i)(2 - i)$.

Solution. (12) By definition of complex number addition,
$(4 + 5i) + (-8 + 2i) = (4 + (-8)) + (5 + 2)i = \boxed{-4 + 7i}$. □

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$(5 + 3i)(2 - i) = (5)(2) + (5)(-i) + (3i)(2) + (3i)(-i) =$
$10 - 5i + 6i - 3i^2 = 10 - 5i + 6i - 3(-1) = \boxed{13 + i}$. □

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Proof. Let $z = a + bi$ and $w = c + di$ in standard form. Then by the definition of multiplication of complex numbers, $\overline{zw} = \overline{(a+bi)(c+di)} = \overline{(a)(c) + (a)(di) + (bi)(c) + (bi)(di)} = \overline{(ac + adi + bci + bdi^2)} = \overline{(ac - bd) + (ad + bc)i} = (ac - bd) - (ad + bc)i$.

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Similarly, $\overline{z}\,\overline{w} = \overline{a + bi}\,\overline{c + di} = (a - bi)(c - di) = (a)(c) + (a)(-di) + (-bi)(c) + (-bi)(-di) = ac - adi - bci + bdi^2 = (ac - bd) - (ad + bc)i$.

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Page A60 Number 26. Perform the indicated operation, and write each expression in the standard form $a + bi$. **(26)** $\dfrac{13}{5 - 12i}$.

Solution. (26) We make the denominator real by multiplying by a version of 1 that involves the conjugate of the denominator:

$$\frac{13}{5-12i} = \frac{13}{5-12i}(1) = \frac{13}{5-12i}\left(\frac{5+12i}{5+12i}\right) = \frac{13(5+12i)}{(5-12i)(5+12i)}$$

$$= \frac{13(5+12i)}{5^2+12^2} = \frac{13(5+12i)}{169} = \frac{5+12i}{13} = \boxed{\frac{5}{13} + \frac{12}{13}i}.$$

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Solution. (30) We make the denominator real by multiplying by a version of 1 that involves the conjugate of the denominator:

$$\frac{2+3i}{1-i} = \frac{2+3i}{1-i}(1) = \frac{2+3i}{1-i}\left(\frac{1+i}{1+i}\right) = \frac{(2+3i)(1+i)}{(1-i)(1+i)}$$

$$= \frac{(2+3i)(1+i)}{1^2+1^2} = \frac{(2+3i)(1+i)}{2} = \frac{(2)(1)+(2)(i)+(3i)(1)+(3i)(i)}{2}$$

$$= \frac{2+2i+3i+3(-1)}{2} = \frac{-1+5i}{2} = \boxed{\frac{-1}{2}+\frac{5}{2}i}.$$

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Solution. (36) Since $i^4 = 1$, we have

$$i^{14} = (i^4)(i^4)(i^4)(i^2) = (1)(1)(1)(-1) = \boxed{-1}.$$

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(38) Since $i^4 = 1$ then $i^{20} = (i^4)^5 = 1^5 = 1$, then $i^{23} = (i^{20})(i^3) = (1)(-i) = -i$. So

$$i^{-23} = \frac{1}{i^{23}} = \frac{1}{-i} = \frac{1}{-i}\left(\frac{i}{i}\right) = \frac{i}{-i^2} = \frac{i}{1} = \boxed{i}.$$

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Solution (continued. (48) Since $i^4 = 1$ then we can reduce the exponents of i by multiples of 4 to get

$$i^7 + i^5 + i^3 + i = (i^4)(i^3) + (i^4)(i) + i^3 + i$$

$$= i^3 + i + i^3 + i = 2i^3 + 2i = 2(-i) + 2i = \boxed{0}.$$

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Page A61 Number 60. Solve in the complex number system $x^2 + 4x + 8 = 0$.

Solution. We use the quadratic formula (which involves the principal square root), $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $a = 1$, $b = 4$, and $c = 8$. So we have

$$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(8)}}{2(1)} = \frac{-4 \pm \sqrt{16 - 32}}{2}$$

$$= \frac{-4 \pm \sqrt{-16}}{2} = \frac{-4 \pm 4i}{2} = -2 \pm 2i.$$

So the solutions are $\boxed{x = -2 - 2i \text{ and } x = -2 + 2i}$, or the solution set is $\boxed{\{-2 - 2i, -2 + 2i\}}$. □

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Solution. The expression $x^3 + 27 = x^3 + 3^3$ is a "Sum of Two Cubes," so from the formula $(x + a)(x^2 - ax + a^2) = x^3 + a^3$ (see A.3. Polynomials) we have $x^3 + 27 = (x + 3)(x^2 - 3x + 9)$. So we see that $x = -3$ is a solution.

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$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)} = \frac{3 \pm \sqrt{9 - 36}}{2}$$

$$= \frac{3 \pm \sqrt{-27}}{2} = \frac{3 \pm \sqrt{9(-3)}}{2} = \frac{3 \pm 3\sqrt{-3}}{2} = \frac{3 \pm 3\sqrt{3}i}{2} = \frac{3}{2} \pm \frac{3\sqrt{3}}{2} i.$$

So the solutions are $\boxed{x = -3,\ x = 3/2 + (3\sqrt{3}/2)i, \text{ and } x = 3/2 - (3\sqrt{3}/2)i}$ and the solution set is $\boxed{\{-3, 3/2 + (3\sqrt{3}/2)i, 3/2 - (3\sqrt{3}/2)i\}}$. □

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Solution. We rewrite the equation as $x^2 - 2x + 6 = 0$ and consider the discriminant $b^2 - 4ac$ where $a = 1$, $b = -2$, and $c = 6$. We have $b^2 - 4ac = (-2)^2 - 4(1)(6) = 4 - 24 = -20$. Since $b^2 - 4ac = -20 < 0$ then the equation has $\boxed{\text{two complex solutions}}$ that are not real (and are conjugates of each other). □

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