Exercise 1.2.85 (Lemniscate) For a nonzero constant a, find the intercepts of $(x^2+y^2)^2 = a^2(x^2-y^2)$. Then test for symmetry with respect to the x-axis, the y-axis, and the origin.

Solution

To find the x-intercepts, we set $y=0$ to get $(x^2+(0)^2)^2 = a^2(x^2-(0)^2)$ or $x^4 = a^2x^2$ or $x^4-a^2x^2 = x^2(x^2-a^2)$. So (by the Zero-Product Property) either $x^2=0$ or $x^2-a^2=(x-a)(x+a)=0$. That is, either $x=0$ or $x=\pm a$. So the $\boxed{x\text{-intercepts are } (0,0),\ (0,-a), \text{ and } (0,a)}$.

To find the y-intercepts, we set $x=0$ to get $((0)^2+y^2)^2 = a^2((0)^2-y^2)$ or $y^4=-a^2y^2$ or $y^4+a^2y^2=0$ or $y^2(y^2+a^2)=0$. Again by the Zero-Product Property, either $y^2=0$ or $y^2+a^2=0$. So $y=0$ (notice that since $a\neq 0$ then $y^2+a^2\neq 0$). So the $\boxed{y\text{-intercept is } (0,0)}$.

To test for symmetry with respect to the x-axis, we replace y with $-y$ to get $(x^2+(-y)^2)^2 = a^2(x^2-(-y)^2)$ or $(x^2+y^2)^2 = a^2(x^2-y^2)$ which is the original equation. So the equation $\boxed{\text{is symmetric WRT the } x\text{-axis}}$.

1.2.85 continued

To test for symmetry with respect to the y-axis, we replace x with $-x$ to get

$$((-x)^2 + y^2)^2 = a^2((-x)^2 - y^2) \quad \text{or}$$

$$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$ which is the original equation. So the equation $\boxed{\text{is symmetric WRT the y-axis}}$.

To test for symmetry with respect to the origin, we replace x with $-x$ and replace y with $-y$ to get

$$((-x)^2 + (-y)^2)^2 = a^2((-x)^2 - (-y)^2) \quad \text{or}$$

$$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$ which is the original equation. So the equation $\boxed{\text{is symmetric WRT the origin}}$. $\square$