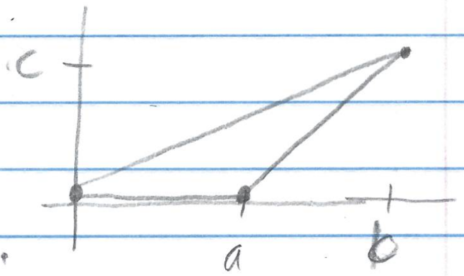
Exercise 1.3.137 Form a triangle using the points $(0,0)$, $(a,0)$, and (b,c), where $a>0$, $b>0$, and $c>0$. Find the point of intersection of the three lines joining the midpoint of a side of the triangle to the opposite vertex.

Solution

Schematically, the triangle is:



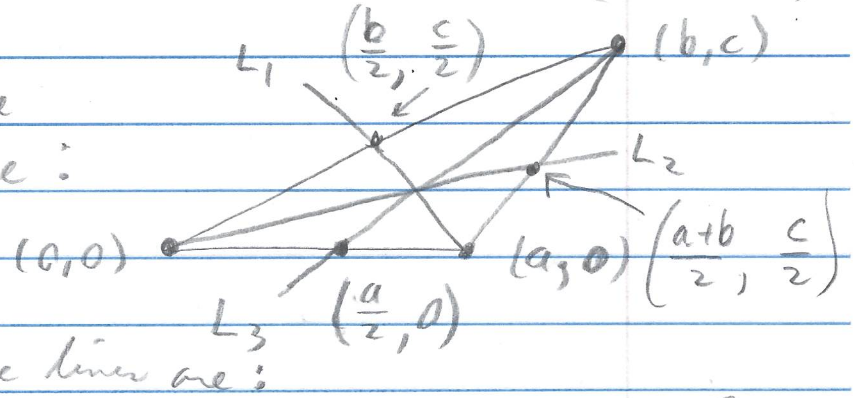
We find the midpoints:

Midpoint between $(0,0)$ and $(a,0)$ $\to \left(\frac{a+0}{2}, \frac{0+0}{2}\right) = \left(\frac{a}{2}, \frac{0}{2}\right)$

midpoint between $(0,0)$ and (b,c) $\to \left(\frac{b+0}{2}, \frac{c+0}{2}\right) = \left(\frac{b}{2}, \frac{c}{2}\right)$

Midpoint between $(a,0)$ and (b,c) $\to \left(\frac{a+b}{2}, \frac{0+c}{2}\right) = \left(\frac{a+b}{2}, \frac{c}{2}\right)$.

The lines we are interested in are:



The slopes of these lines are:

$$m_1 = \frac{c/2-0}{b/2-a} = \frac{c}{b-2a}, \quad m_2 = \frac{c/2-0}{(a+b)/2-0} = \frac{c}{a+b}$$

and $$m_3 = \frac{c-0}{b-a/2} = \frac{2c}{2b-a}.$$

1.3.137 continued

The equations of the lines from the point slope formula is:

L_1: $y-(0)=\frac{c}{b-2a}(x-a)$ or $y=\frac{c}{b-2a}(x-a)$,

L_2: $y-(0)=\frac{c}{a+b}(x-(0))$ or $y=\frac{c}{a+b}x$,

L_3: $y-(c)=\frac{2c}{2b-a}(x-(b))$

or $y-c=\frac{2c}{2b-a}(x-b)$.

Lines L_1 and L_2 intersect when the y coordinates are the same, or when

$$\frac{c}{b-2a}(x-a)=\frac{c}{a+b}x \text{ or } \frac{cx-ac}{b-2a}=\frac{cx}{a+b}$$

or (cross multiplying)

$(a+b)(cx-ac)=(b-2a)cx$ or

$(a+b)cx-(b-2a)cx=(a+b)ac$ or

$x(ac+bc-bc+2ac)=(a+b)ac$ or

$$x=\frac{(a+b)ac}{ac+2ac}=\frac{(a+b)ac}{3ac}=\frac{a+b}{3}.$$

The corresponding y-value (based on line L_2) is $y=\frac{c}{a+b}\left(\frac{a+b}{3}\right)=\frac{c}{3}$.

Hence L_1 and L_2 intersect at $\left(\frac{a+b}{3}, \frac{c}{3}\right)$.

1.3.137 continued again

We now test to see the point $\left(\frac{a+b}{3}, \frac{c}{3}\right)$ is on line L_3:

$$\left(\frac{c}{3}\right) - c = \frac{2c}{2b-a}\left(\left(\frac{a+b}{3}\right) - b\right) \text{ or}$$

$$-\frac{2c}{3} = \frac{2c}{2b-a}\left(\frac{a-2b}{3}\right) = \frac{2c}{3}(-1) = \frac{-2c}{3},$$

and so the point does lie on L_3.
So the point of intersection of lines L_1, L_2, and L_3 is $\left(\frac{a+b}{3}, \frac{c}{3}\right)$. □